

Holography for Einstein-Maxwell-dilaton theories from generalized dimensional reduction

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ABSTRACT: We show that a class of Einstein-Maxwell-Dilaton (EMD) theories are related to higher dimensional AdS-Maxwell gravity via a dimensional reduction over compact Einstein spaces combined with continuation in the dimension of the compact space to non-integral values ('generalized dimensional reduction'). This relates (fairly complicated) black hole solutions of EMD theories to simple black hole/brane solutions of AdS-Maxwell gravity and explains their properties. The generalized dimensional reduction is used to infer the holographic dictionary and the hydrodynamic behavior for this class of theories from those of AdS. As a specific example, we analyze the case of a black brane carrying a wave whose universal sector is described by gravity coupled to a Maxwell field and two neutral scalars. At thermal equilibrium and finite chemical potential the two operators dual to the bulk scalar fields acquire expectation values characterizing the breaking of conformal and generalized conformal invariance. We compute holographically the first order transport coefficients (conductivity, shear and bulk viscosity) for this system.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Black Holes, Holography and condensed matter physics (AdS/CMT).

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1. Introduction

Recently there has been increased interest in understanding holography for Einstein-Maxwell-dilaton theories. Such theories have the right field content to describe holographically systems at finite charge density, possibly in the presence of condensates, and as such they have appeared in the holographic modeling of strongly interacting condensed matter systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], see [15] for a recent review and further references. Since many of the relevant solutions are not asymptotically AdS, it is not a priori clear how to set up holography. It is the purpose of this paper to provide a holographic dictionary for a class of such theories.

In order to set up holography one needs to understand the asymptotic structure of the field equations, identify where the source of the dual operator is located, evaluate the on-shell action, determine a set of (local covariant) counterterms and finally compute the renormalized 1-point functions in the presence of sources (see [16, 17] for reviews). This

procedure has been carried out for AdS gravity (coupled to matter fields) in [18] (see also [19, 20]) and for gravity coupled to a scalar field with exponential potential in [21, 22]. The latter case is associated with the near-horizon limit of the non-conformal branes [23, 24]. It was realized later in [25] that the two cases are actually closely related: one can obtain the results in [22] from those in [18] *via* a ‘generalized dimensional reduction’.

More precisely, it was shown in [22] that the $(d+1)$ -dimensional gravity-dilaton system with an exponential potential $V(\phi) \sim \exp(-\delta\phi)$ can be obtained from $\text{AdS}_{2\sigma+1}$ gravity by diagonal reduction over $T^{2\sigma-d}$ torus. This is a consistent reduction, so the structure of the solutions of the field equations of the reduced theory and all other results needed in order to set up holography can be deduced from that of AdS gravity. The results depend smoothly on σ as long as $\sigma > d/2$. This can be seen by inspection of the results but it is also intuitively clear: the dimension of the torus, $(2\sigma - d)$, should be positive. It follows that one can use the dimensional reduction in order to establish a holographic dictionary for this theory when¹ $\sigma > d/2$ (this translates into a constraint on the slope of the potential, $\delta^2 < 2/(d-1)$). Indeed, one can check that the results established in [22] by a direct analysis of the field equations etc. are reproduced exactly. This method was also applied successfully to probe non-conformal branes [26]: the corresponding holographic dictionary was obtained from the results in [27] in this way.

This method can be used in order to set up holography for any theory that is related to a theory for which the holographic dictionary is known *via* such a ‘generalized consistent reduction’. The upstairs theory can thus be AdS gravity coupled to general matter (scalar fields, fermions, gauge fields, form fields). The reduction must be consistent, *i.e.* all solutions of the lower-dimensional theory should be solutions of the higher-dimensional theory. This is necessary in order to be able to deduce the structure of the field equations of the lower-dimensional theory etc. from that of the higher-dimensional theory. The reduction will be ‘generalized’ if the reduced theory depends smoothly on the dimension of the compactification manifold (and perhaps other such parameters) which could then be continued to be any real number.

In this paper we will focus on a lower-dimensional theory that contains Maxwell fields. One way to obtain Maxwell fields is to replace the diagonal torus reduction by a general non-diagonal reduction and another is to have Maxwell fields already in the upstairs theory. The diagonal torus reduction can also be replaced by a reduction over an Einstein manifold (which is also consistent as long as we only keep the mode parametrizing the overall size of the compact manifold). Such reduction produces a lower-dimensional theory with a potential having two exponential terms. Yet another possibility is to start with form fields in higher dimensions. This case has been analyzed in [14] and it will not be discussed here.

For the applications of interest, one would like to have explicit black hole solutions where the scalar and Maxwell fields are non-trivial. It turns out that the theories obtained *via* a generalized dimensional reduction from a higher-dimensional AdS gravity (possibly coupled to a Maxwell field) are the same as the theories where non-extremal black hole solutions are explicitly known. The asymptotics of these solutions are often ‘unconventional’

¹The non-conformal branes correspond to specific rational values of σ .

as the scalar field blows up at the infinity. This behavior complicates the computation of conserved charges. Our results streamline this discussion as well: conserved charges can be computed using the holographic stress energy tensor and the holographic conserved current. Since these objects originate from their higher-dimensional AdS counterparts, they satisfy all expected thermodynamic identities, just as the AdS ones, [28]. The solutions themselves (which often look complicated) originate from simpler solutions in higher dimensions. Moreover, one can also obtain a description of the hydrodynamic regime from that of the higher-dimensional AdS case. This leads to the derivation of the relevant transport coefficients and can explain some special relations they may satisfy. For example, these considerations explained in [25] that the special value of the bulk to shear viscosity ratio for all backgrounds which asymptote to the non-conformal brane background is due to the conformal symmetry of the higher-dimensional theory. The same method was also used in [29] in order to compute transport coefficients for the Quark-Gluon Plasma using a holographic model of QCD.

This paper is organized as follows. In the next section, we list the cases where the Einstein-Maxwell-dilaton theory can be oxidised to a higher-dimensional AdS-Maxwell theory and we discuss their black hole solutions, their uplift to AdS black holes and how their charges and thermodynamics can be explained *via* the lift to higher dimensions. Then, in section 3 we specialize to one of the relevant cases and we fully carry out the program discussed above. Finally, in appendix A we prove that the dimensional reductions used in this paper are consistent.

2. Oxidation of Einstein-Maxwell-Dilaton theories

In this section we will consider how higher-dimensional AdS(-Maxwell) gravity reduces to Einstein-Maxwell-Dilaton (EMD) theories *via* a (generalized) consistent (non-)diagonal Kaluza-Klein reduction. We will further connect the properties of the (EMD) black hole solutions with those of higher-dimensional black holes, charged (asymptotically flat) black p -branes and boosted black branes.

The higher-dimensional action is given by

$$S_{(2\sigma+1)} = \frac{1}{16\pi G_N^{(2\sigma+1)}} \int_{\mathcal{M}} d^M x d^a y \sqrt{-g_{(2\sigma+1)}} \left[R_{(2\sigma+1)} - \frac{1}{4} F^2 - 2\Lambda \right] \quad (2.0.1)$$

The integral is over the bulk $(2\sigma + 1)$ -dimensional spacetime \mathcal{M} . Capital latin indices M, N, \dots run from 0 to d , and denote lower, $(d + 1)$ -dimensional spacetime coordinates, while lowercase latin indices a, b, \dots will typically run from $d + 1$ to 2σ and denote internal coordinates. Higher case latin indices A, B, \dots refer to the higher-dimensional spacetime coordinates and run from 0 to 2σ . Lowercase greek indices μ, ν, \dots are higher-dimensional indices and run from 0 to $2\sigma - 1$, while lowercase latin indices i, j, \dots are lower-dimensional boundary indices and run from 0 to $d - 1$. The Maxwell terms with straight latin uppercase originate from higher dimensions, $A = A_A dx^A$, with field strength

$$F = \frac{1}{2} F_{AB} dx^A \wedge dx^B = dA, \quad F_{AB} = 2\partial_{[A} A_{B]}, \quad (2.0.2)$$

where the brackets are the usual antisymmetry operation. The Maxwell term is invariant upon taking the Hodge dual $\star F$ of the field strength, which allows to generate magnetic solutions from electric solutions and vice-versa.

In what follows we will be interested in Kaluza-Klein reductions on a $(2\sigma-d)$ -dimensional internal space $\mathbf{X}_{(2\sigma-d)}$ times a $(d+1)$ -dimensional manifold $\mathcal{M}_{(d+1)}$; details of the compactifications are given in Appendix A. The reductions of interest are over Einstein manifolds. Recall that a p -dimensional Einstein manifold $\mathbf{X}_{(p)}$ satisfies

$$R_{ab}^{(p)} = (p-1)\lambda_{(p)}g_{ab}. \quad (2.0.3)$$

We will denote the metric of $\mathbf{X}_{(p)}$ by $ds^2 = dX_{(p)}^2$ and its volume by $V_{(p)}$. When the Einstein manifold is homogeneous, this implies that the Riemann tensor is

$$R_{abcd}^{(p)} = \lambda_{(p)}(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad (2.0.4)$$

where $\lambda_{(p)}$ is then normalized to $\pm 1, 0$. We also define the Anti-de Sitter radius in $2\sigma+1$ dimensions as:

$$-2\Lambda\ell_{(2\sigma+1)}^2 = 2\sigma(2\sigma-1). \quad (2.0.5)$$

2.1 Diagonal reduction to Einstein-Dilaton theories

Let us start from the AdS-Einstein action in $2\sigma+1$ dimensions:

$$S_{(2\sigma+1)} = \frac{1}{16\pi G_N^{(2\sigma+1)}} \int_{\mathcal{M}} d^{2\sigma+1}x \sqrt{-g_{(2\sigma+1)}} [R - 2\Lambda]. \quad (2.1.1)$$

We show in appendix A that the reduction ansatz

$$ds_{(2\sigma+1)}^2 = e^{-\delta_1\phi} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta_1}(\delta_c^2 - \delta_1^2)} dX_{(2\sigma-d)}^2 \quad (2.1.2)$$

with

$$\delta_1^2 = \frac{2(2\sigma-d)}{(d-1)(2\sigma-1)} \leq \delta_c^2 \equiv \frac{2}{(d-1)} \iff 2\sigma = \frac{d - \frac{d-1}{2}\delta_1^2}{1 - \frac{d-1}{2}\delta_1^2} \geq 0, \quad (2.1.3)$$

consistently reduces (2.1.1) to a $(d+1)$ -dimensional theory with action

$$S_{(d+1)} = \frac{1}{16\pi G_N^{(d+1)}} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g_{(d+1)}} \left[R - \frac{1}{2}(\partial\phi)^2 - 2\Lambda_1 e^{-\delta_1\phi} - 2\Lambda_2 e^{-\delta_2\phi} \right], \quad (2.1.4)$$

where

$$\Lambda = \Lambda_1, \quad R_{(2\sigma-d)} = -2\Lambda_2, \quad \delta_2 = \frac{\delta_c^2}{\delta_1} \geq \delta_c. \quad (2.1.5)$$

Note that consistency of the reduction requires that $\mathbf{X}_{(2\sigma-d)}$ is an Einstein manifold.

The Einstein-Dilaton theory we obtain has a scalar potential comprising two exponential terms, whose origin are respectively the higher-dimensional cosmological constant, Λ , and the curvature $R_{(2\sigma-d)}$ of the internal space. Note that in this case, the slope of the exponential δ_1 is restricted to the interval $[0, \delta_c]$ so that the number of reduced dimensions

$2\sigma - d$ remains positive. Consequently, the slope of the exponential δ_2 is restricted to the complementary interval $[\delta_c, +\infty]$.

Since the action (2.1.4) is invariant under the exchange $\{\Lambda_1, \delta_1\} \leftrightarrow \{\Lambda_2, \delta_2\}$, it may also be obtained from the Einstein-AdS action (2.1.1) by the alternate reduction ansatz,

$$ds_{(2\sigma+1)}^2 = e^{-\frac{\delta_c^2}{\delta_1} \phi} ds_{(d+1)}^2 + e^{(\delta_1^2 - \delta_c^2) \frac{\phi}{\delta_1}} dX_{(2\sigma-d)}^2, \quad (2.1.6)$$

with

$$\delta_c^2 < \delta_1^2 = \frac{2(2\sigma - 1)}{(d - 1)(2\sigma - d)} < \delta_{max}^2 \equiv \frac{2d}{(d - 1)} \iff +\infty > 2\sigma - d > 1, \quad (2.1.7)$$

$$\Lambda = \Lambda_2, \quad R_{(2\sigma-d)} = -2\Lambda_1, \quad \delta_2 = \frac{\delta_c^2}{\delta_1} \leq \delta_c^2, \quad (2.1.8)$$

Note that the upper bound on the value of δ_1 corresponds to the internal space being one-dimensional, *i.e.* $R_{(1)} = -2\Lambda_1 = 0$.

Since σ is related to the dimension of the higher-dimensional theory *via* $D = 2\sigma + 1$, it should be a (half) integer. However, after reduction σ enters algebraically as a parameter in the $(d + 1)$ -dimensional action, so one may analytically continue its value to any real number [25], modulo restrictions that arise from the requirement that the lower-dimensional theory is well-behaved (kinetic terms should be positive definite etc.). This generates the continuous family of theories (2.1.4) labeled by a real parameter δ_1 , related to a higher-dimensional AdS-Mawxell theory *via* generalized dimensional reduction.

Analyzing the equations of motion derived from action (2.1.4), one can show that there exist analytic black hole solutions precisely when the theory is related to a higher-dimensional AdS theory, namely when $\delta_2 = \delta_c^2/\delta_1$. The solution is given by, [30, 31, 32, 33],

$$ds_{(d+1)}^2 = -V(r)dt^2 + \frac{e^{\delta_1 \phi} dr^2}{V(r)} + r^2 dX_{(d-1)}^2, \quad (2.1.9)$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 + \frac{(d-2)\lambda_{(d-1)}r^{(d-1)\delta_1^2}}{\left(1 - \frac{d-1}{2}\delta_1^2\right)\left(d - 2 + \frac{d-1}{2}\delta_1^2\right)} - mr^{2-d+\frac{d-1}{2}\delta_1^2}, \quad (2.1.10)$$

$$e^\phi = r^{(d-1)\delta_1} \quad (2.1.11)$$

$$-2\Lambda_2 = \frac{(d-1)^2(d-2)\delta_1^2\lambda_{(d-1)}}{2\left(1 - \frac{d-1}{2}\delta_1^2\right)} \quad (2.1.12)$$

$$-2\Lambda_1\ell^2 = (d-1)\left(d - \frac{d-1}{2}\delta_1^2\right). \quad (2.1.13)$$

The solution has a curvature singularity at $r = 0$ and an event horizon wherever $V(r_+) = 0$. One may set $\lambda_{(d-1)} = 0$ in the above expression and obtain the generic neutral planar black hole solution, whose horizon has topology \mathbf{R}^{d-1} . In this case the scalar potential reduces to a single exponential. At first sight it may seem that the solution is singular in the limit $\delta^2 \rightarrow \delta_c^2$, but a scaling limit can be taken if one simultaneously sends $\lambda_{(p-1)} \rightarrow 0$ while keeping the ratio of the two previous quantities fixed.

The action (2.1.4) is symmetric under the exchange

$$\Lambda_1 \longleftrightarrow \Lambda_2, \quad \delta_1 \longleftrightarrow \delta_2, \quad (2.1.14)$$

which allows to generate a solution dual to (2.1.9).

Thermodynamics. The thermodynamics of these black holes may be calculated by computing the Euclidean on-shell action and taking appropriate derivatives of the thermodynamic potential with respect to the thermodynamic variables, *e.g.*:

$$S[T] = -\frac{dF[T]}{dT}, \quad M[T] = F[T] - T\frac{dF[T]}{dT}, \quad F[T] = M[T] - T S[T]. \quad (2.1.15)$$

in the canonical ensemble.

These results may be put on firmer footing. As we discuss in detail in section 3, the generalized dimensional reduction leads to a holographic stress energy tensor, which may be used to compute the various thermodynamic quantities (such as the mass of the black hole). Either way, these computations lead to the results:

$$T_{(d+1)} = \frac{r_+^{1-\frac{d-1}{2}\delta_1^2}}{4\pi} \left[\left(d - (d-1)\frac{\delta_1^2}{2} \right) \ell^{-2} + \frac{(d-2)\lambda_{(d-1)}}{\left(1 - \frac{d-1}{2}\delta_1^2\right)} r_+^{(d-1)\delta_1^2-2} \right], \quad (2.1.16)$$

$$S_{(d+1)} = \frac{V_{(d-1)}}{4G_N^{(d+1)}} r_+^{(d-1)}, \quad (2.1.17)$$

$$M_{(d+1)} = \frac{V_{(d-1)}(d-1)r_+^{d-\frac{d-1}{2}\delta_1^2}}{16\pi G_N^{(d+1)}} \left[\ell^{-2} + \frac{(d-2)\lambda_{(d-1)}r_+^{(d-1)\delta_1^2-2}}{\left(1 - \frac{d-1}{2}\delta_1^2\right) \left(d-2 + \frac{d-1}{2}\delta_1^2\right)} \right]. \quad (2.1.18)$$

$V_{(d-1)}$ stands for the volume of $\mathbf{X}_{(d-1)}$. One may check that the first law holds:

$$dM = TdS \quad \Leftrightarrow \quad dF = -SdT, \quad (2.1.19)$$

and then examine global and local thermodynamical equilibrium by computing the free energy and the heat capacity:

$$F_{(d+1)} = \frac{V_{(d-1)} \left(1 - \frac{d-1}{2}\delta_1^2\right) r_+^{d-\frac{d-1}{2}\delta_1^2}}{16\pi G_N^{(d+1)}} \left[-\ell^2 + \frac{(d-2)\lambda_{(d-1)}r_+^{(d-1)\delta_1^2-2}}{\left(1 - \frac{d-1}{2}\delta_1^2\right) \left(d-2 + \frac{d-1}{2}\delta_1^2\right)} \right] \quad (2.1.20)$$

$$C = T \frac{dS}{dT}. \quad (2.1.21)$$

We will shortly see that these results descend from higher dimensions.

The Einstein dilaton action with two exponential terms in its potential was already recognized as descending from a higher-dimensional Einstein action by the authors of [34, 35], but they did not consider the neutral, dilatonic solution 2.1.9 from the point of view of the higher-dimensional theory, while the authors of [36] only considered the oxidation of (2.1.9) with a single potential turned on, and also not in the context of 'generalized' reductions.

From the previous considerations and given a specific exponential scalar potential

$$V(\phi) = V_0 e^{-\delta_1 \phi}, \quad (2.1.22)$$

we now understand it can descend from a higher-dimensional theory in two ways, as explained in more details in Appendix A.1. If $\delta_1^2 \leq \delta_c^2$, V_0 may be identified with a higher-dimensional cosmological constant, while if $\delta_c^2 \leq \delta_1^2 < \delta_{max}^2$, it may be identified with the

curvature of the internal space $\mathbf{X}_{(2\sigma-d)}$ over which the reduction is performed. $\delta_1^2 = \delta_c^2$ corresponds to an infinite number of dimensions, while $\delta_1^2 = \delta_{max}^2$ to a single one so that the reduction over $\mathbf{X}_{(1)}$ does not generate a potential.

In the case where the theory has a single potential and for planar black holes, it was shown in [37, 38, 6] that for $\delta_1^2 \leq \delta_c^2$, the spectrum of fluctuations was continuous and gapless, while for $\delta_1^2 > \delta_c^2$, it was discrete with a gap. Moreover, this gives a recipe for generating first-order phase transitions in EMD theories, while imposing a planar boundary: by considering a potential with two exponentials, with slopes verifying $\delta_1 \leq \delta_c$ and $\delta_2 > \delta_c$, the former should act as a cosmological constant, the latter as horizon curvature. This matches with the intuition from the KK reduction and was exhibited in $d = 4$ in [38].

We will now consider the uplift of the solution (2.1.9)-(2.1.13) for the two different ranges of δ_1 . As we will see they originate from different higher-dimensional spacetimes in the two respective cases.

Oxidation for $\delta_1^2 \leq \delta_c^2$: In this case we should use the oxidation ansatz (2.1.2) with δ_1 given by (2.1.3). The uplift of the solution (2.1.9)-(2.1.13) is then

$$ds_{(2\sigma+1)}^2 = -\frac{f(\rho)}{\rho \ell_{(2\sigma+1)}^2} d\tau^2 + \frac{\ell_{(2\sigma+1)}^2 d\rho^2}{4\rho^2 f(\rho)} + \rho^{-1} \left(dX_{(d-1)}^2 + dX_{(2\sigma-d)}^2 \right), \quad (2.1.23)$$

$$f(\rho) = 1 + \ell_{(2\sigma+1)}^2 (\lambda_{(2\sigma-1)} \rho - m \rho^\sigma). \quad (2.1.24)$$

To obtain this result, we have used the change of coordinates:

$$r^{1-\frac{d-1}{2}\delta_1^2} = r^{\frac{d-1}{2\sigma-1}} = \rho^{-\frac{1}{2}}, \quad \tau = \frac{2\sigma-1}{d-1} t, \quad (2.1.25)$$

and normalised the curvature on the horizon as

$$(2\sigma-2)\lambda_{(2\sigma-1)} = (d-2)\lambda_{(d-1)} = (2\sigma-d-1)\lambda_{(2\sigma-d)} = \frac{-2\Lambda_2}{2\sigma-d}. \quad (2.1.26)$$

The relation (2.1.12) can now be understood from the higher-dimensional perspective as necessary for the space $\mathbf{X}_{(d-1)} \times \mathbf{X}_{(2\sigma-d)}$ to solve the higher-dimensional Einstein equations. The uplifted spacetime is then simply the Schwarzschild-AdS $_{(2\sigma+1)}$ black hole, where the horizon topology is not $\mathbf{X}_{(2\sigma-1)}$ but the product space $\mathbf{X}_{(d-1)} \times \mathbf{X}_{(2\sigma-d)}$. Their normalised curvatures $\lambda_{(2\sigma-d)}$, $\lambda_{(d-1)}$ must satisfy (2.1.26): as a consequence, only one of the λ may generically be set to $\pm 1, 0$, except if $2\sigma-d = d-1$ (identical compact spaces). For spheres, this means they cannot have the same radius. As remarked in the previous section, the horizon curvature can be set to zero, in which case $\Lambda_2 = 0$ and the AdS planar black hole is recovered. In higher-dimensional Einstein gravity, the requirement that the horizon is homogeneous is relaxed to being simply Einstein: this is essential to our ability to carry out the generalized reduction of higher-dimensional solutions in order to generate lower-dimensional ones.

The thermodynamics of asymptotically (locally) AdS spaces can be worked out using standard holographic technology [28]. In particular, the holographic stress-energy tensor can be computed using (3.2.7): the knowledge of the σ -th term in the Fefferman-Graham

expansion is enough when the spacetime has a flat boundary. For curved boundaries, one needs to include additional terms (see [39] for a review). Let us work out in more detail the case for $\sigma = 2$, that is a Schwarzschild black hole in AdS_5 with boundary $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^2$. The general formula reads [18]:

$$\langle T_{\mu\nu} \rangle_{2\sigma} = \frac{4V_{(3)}}{16\pi G_N^{(5)} \ell_{(5)}^2} \left[g_{(4)\mu\nu} - \frac{g_{(2)\mu\nu}}{8} \left((\text{Tr} g_{(2)})^2 - \text{Tr}(g_{(2)}^2) \right) - \frac{1}{2} g_{(2)\mu\nu}^2 + \frac{\text{Tr} g_{(2)}}{4} g_{(2)\mu\nu} \right] \quad (2.1.27)$$

while from (2.1.23)

$$g_{(0)} = \text{Diag}(-1, 1, 1, \sin^2 \theta), \quad (2.1.28)$$

$$g_{(2)} = \frac{\ell_{(5)}^2 \lambda_{(3)}}{2} \text{Diag}(1, 1, 1, \sin^2 \theta), \quad (2.1.29)$$

$$g_{(4)} = \frac{\ell_{(5)}^4 \lambda_{(3)}^2}{4} g_{(0)} + \frac{\ell_{(5)}^2 m}{4} \text{Diag}(3, 1, 1, \sin^2 \theta), \quad (2.1.30)$$

so that

$$\langle T_{\mu\nu} \rangle_{2\sigma} = \frac{V_{(3)} m}{16\pi G_N^{(5)}} [4\delta_{\mu 0} \delta_{\nu 0} + \eta_{\mu\nu}]. \quad (2.1.31)$$

Generalizing to arbitrary dimension, one finds for the spacetime (2.1.23):

$$T_{(2\sigma+1)} = \frac{1}{4\pi \rho_+^{\frac{1}{2}}} \left[\frac{2\sigma}{\ell_{(2\sigma+1)}^2} + (2\sigma - 2) \lambda_{(2\sigma-1)} \rho_+ \right] \quad (2.1.32)$$

$$S_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{4G_N^{(2\sigma+1)}} \rho_+^{\frac{1}{2}-\sigma} \quad (2.1.33)$$

$$M_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{16\pi G_N^{(2\sigma+1)}} (2\sigma - 1) m \quad (2.1.34)$$

$$F_{(2\sigma+1)} = -\frac{V_{(2\sigma-1)} \rho_+^{-\sigma}}{16\pi G_N^{(2\sigma+1)}} \left[\ell_{(2\sigma+1)}^2 - \lambda_{(2\sigma-1)} \rho_+ \right] \quad (2.1.35)$$

which coincide with the expressions in [40, 41].

We can now check the validity of the thermodynamic formulæ (2.1.16)-(2.1.20), taking care of including constant factors due to the change of coordinates (2.1.25). Indeed, inspecting the reduction Ansatz and the definition of the thermodynamic potential from the on-shell Euclidean action, one may show that

$$\beta_{(2\sigma+1)} = \frac{2\sigma - 1}{d - 1} \beta_{(d+1)}, \quad S_{(2\sigma+1)} = S_{(d+1)} \quad (2.1.36)$$

$$\beta_{(2\sigma+1)} F_{(2\sigma+1)} = \beta_{(d+1)} F_{(d+1)}, \quad \beta_{(2\sigma+1)} M_{(2\sigma+1)} = \beta_{(d+1)} M_{(d+1)} \quad (2.1.37)$$

which in turn lead to (2.1.16)-(2.1.20).

Oxidation for $\delta_c^2 < \delta_1^2 < \delta_{max}^2$: In this case we use the oxidation Ansatz (2.1.6) and δ_1 is given by (2.1.7). The uplift of (2.1.9)-(2.1.13) becomes:

$$ds_{(2\sigma+1)}^2 = -\frac{f(\rho)}{\ell_{(2\sigma-d+2)}^2 \rho} d\tau^2 + \frac{\ell_{(2\sigma-d+2)}^2 d\rho^2}{4\rho^2 f(\rho)} + \rho^{-1} dX_{(2\sigma-d)}^2 + dX_{(d-1)}^2, \quad (2.1.38)$$

$$f(\rho) = 1 + \ell_{(2\sigma-d+2)}^2 \left(\lambda_{(2\sigma-d)} \rho - m \rho^{\frac{1}{2}(2\sigma-d+1)} \right). \quad (2.1.39)$$

with the change of coordinates

$$r^{\frac{d-1}{2}\delta_1^2-1} = r^{\frac{d-1}{2\sigma-d}} = \rho^{-\frac{1}{2}}, \quad \tau = \frac{2\sigma-d}{d-1} t, \quad (2.1.40)$$

and

$$-2\Lambda_2 = -(d-2)(2\sigma-1)\lambda_{(d-1)} = \frac{(2\sigma-1)(2\sigma-d+1)}{\ell_{(2\sigma-d+2)}^2}, \quad -2\Lambda_1 = (2\sigma-d)(2\sigma-d-1)\lambda_{(2\sigma-d)}. \quad (2.1.41)$$

The former stems from (2.1.12), the latter from having exchanged the roles of Λ_1 and Λ_2 in the oxidation. The solution describes an AdS black hole in $(2\sigma-d+2)$ dimensions times a $(d-1)$ -dimensional hyperbolic plane², with topology $\text{AdS}_{2\sigma-d+2} \times \mathbf{X}_{(d-1)}$. Note that if $\lambda_{(d-1)} = 0$, we also need to set $\Lambda_2 = 0$ and the solution (2.1.38) becomes the familiar neutral black $(d-1)$ -brane, where one adds $(d-1)$ flat directions to the Schwarzschild metric.

As in the previous subsection, we can recover the appropriate $2\sigma+1$ behaviours for the thermodynamics of the black $(d-1)$ -brane:

$$T_{(2\sigma+1)} = \frac{1}{4\pi\rho_+^{\frac{1}{2}}} \left[\frac{(2\sigma-d+1)}{\ell_{(2\sigma-d+2)}^2} + (2\sigma-d-1)\lambda_{(2\sigma-d)}\rho_+ \right] \quad (2.1.42)$$

$$S_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{4G_N^{(2\sigma+1)}} \rho_+^{-\frac{1}{2}(2\sigma-d)} \quad (2.1.43)$$

$$M_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{16\pi G_N^{(2\sigma+1)}} (2\sigma-d)m \quad (2.1.44)$$

$$F_{(2\sigma+1)} = -\frac{V_{(2\sigma-1)}\rho_+^{-\frac{1}{2}(2\sigma-d+1)}}{16\pi G_N^{(2\sigma+1)}} \left[\ell_{(2\sigma-d+2)}^2 - \lambda_{(2\sigma-d)}\rho_+ \right]. \quad (2.1.45)$$

The solution dual to (2.1.9) under the exchange (2.1.14) uplifts to (2.1.38) if $\delta_1^2 < \delta_c^2$, and to (2.1.23) if $\delta_c^2 < \delta_1^2 < \delta_{max}^2$.

2.2 Diagonal reduction to Einstein-Maxwell-Dilaton theories

In this subsection, we would like to determine how Einstein-Maxwell theories

$$S_{(2\sigma+1)} = \frac{1}{16\pi G_N^{(2\sigma+1)}} \int_{\mathcal{M}} d^M x d^a y \sqrt{-g_{(2\sigma+1)}} \left[R_{(2\sigma+1)} - \frac{1}{4} F^2 - 2\Lambda \right] \quad (2.2.1)$$

²A well-known way of making the curvature of the brane worldvolume positive is to include a $(d+1)$ -field strength in the action.

can give rise to Einstein-Maxwell-Dilaton theories

$$S_{(d+1)} = \frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-g_{(d+1)}} \left[R - \frac{1}{2} \partial\phi^2 - \frac{1}{4} e^{\gamma\phi} F^2 - 2\Lambda_1 e^{-\delta_1\phi} - 2\Lambda_2 e^{-\delta_2\phi} \right]. \quad (2.2.2)$$

via diagonal Kaluza-Klein reduction; that is, the lower-dimensional Maxwell field originates from a higher-dimensional one³.

We shall consider a reduction Ansatz

$$A_A = (A_M(x^N), 0), \quad (2.2.3)$$

for the Maxwell field, to avoid generating axionic fields in the lower-dimensional theory.

Using the Ansatz (2.1.2) for the metric, (2.2.1) reduces consistently to (2.2.2) with

$$\gamma = \delta_1 < \delta_c, \quad \Lambda_1 = \Lambda, \quad \delta_c < \delta_2 = \frac{2}{(d-1)\delta_1}, \quad -2\Lambda_2 = R_{(2\sigma-d)}, \quad (2.2.4)$$

while using the Ansatz (2.1.6) yields

$$\gamma = \delta_2 = \frac{2}{(d-1)\delta_1} < \delta_c, \quad \Lambda_2 = \Lambda, \quad \delta_c < \delta_1, \quad -2\Lambda_1 = R_{(2\sigma-d)}. \quad (2.2.5)$$

Note that the introduction of the gauge field breaks the duality (2.1.14): in the theory (2.2.4), exchanging $\delta_1 \leftrightarrow \delta_2$ and $\Lambda_1 \leftrightarrow \Lambda_2$ does not map back to (2.2.4) but to (2.2.5), because γ is mapped to δ_2 . This means a single solution of (2.2.2) may not be uplifted to two different solutions of (2.2.1) as in section 2.1. In both reduction schemes, $\gamma < \delta_c$.

2.2.1 Solution with two exponential-potential

The neutral black hole solution (2.1.9) can be generalized to an already known charged solution if one sets $\gamma = \delta_1$ and $\delta_2 = 2/(d-1)\delta_1$. The solution becomes, see [30, 31, 32, 42]:

$$ds_{(d+1)}^2 = -V(r)dt^2 + \frac{e^{\delta_1\phi}dr^2}{V(r)} + r^2 dX_{(d-1)}^2, \quad (2.2.6)$$

$$V(r) = \left(\frac{r}{\ell}\right)^2 + \frac{(d-2)\lambda_{(d-1)}r^{(d-1)\delta_1^2}}{\left(1 - \frac{d-1}{2}\delta_1^2\right)\left(d-2 + \frac{d-1}{2}\delta_1^2\right)} - mr^{2-d+\frac{d-1}{2}\delta_1^2} + q^2r^{-2(d-2)}, \quad (2.2.7)$$

$$e^\phi = r^{(d-1)\delta_1}, \quad (2.2.8)$$

$$A = -\sqrt{\frac{2(d-1)}{d-2 + \frac{d-1}{2}\delta_1^2}} q r^{-(d-2) - \frac{d-1}{2}\delta_1^2} dt, \quad (2.2.9)$$

$$-2\Lambda_2 = \frac{(d-1)^2(d-2)\delta_1^2\lambda_{(d-1)}}{2\left(1 - \frac{d-1}{2}\delta_1^2\right)}, \quad (2.2.10)$$

$$-2\Lambda_1\ell^2 = (d-1)\left(d - \frac{d-1}{2}\delta_1^2\right). \quad (2.2.11)$$

³The case where such a Maxwell field comes from a higher-dimensional p -form potential has been investigated in some details in [14].

Thermodynamics. Using the generalized reduction of the holographic conserved charges, one may compute various thermodynamic quantities, such as the temperature, entropy and mass of the black hole:

$$T_{(d+1)} = \frac{r_+^{1-\frac{d-1}{2}\delta_1^2}}{4\pi\ell^2} \left[\left(d - \frac{d-1}{2}\delta_1^2 \right) + \frac{(d-2)\lambda_{(d-1)}\ell^2}{\left(1 - \frac{d-1}{2}\delta_1^2\right)r_+^{2-(d-1)\delta^2}} - \frac{q^2\ell^2}{r_+^{2(d-1)}} \right], \quad (2.2.12)$$

$$Q_{(d+1)} = \frac{V_{(d-1)}q}{16\pi G_N^{(d-1)}} \sqrt{2(d-1) \left(d - 2 + \frac{d-1}{2}\delta_1^2 \right)} \quad (2.2.13)$$

$$\mu_{(d+1)} = \sqrt{\frac{2(d-1)}{(d-2 + \frac{d-1}{2}\delta_1^2)}} q r_+^{2-d-\frac{d-1}{2}\delta_1^2}, \quad (2.2.14)$$

$$S_{(d+1)} = \frac{V_{(d-1)}}{4G_N^{(d-1)}} r_+^{(d-1)}, \quad (2.2.15)$$

$$M_{(d+1)} = \frac{V_{(d-1)}(d-1)}{16\pi G_N^{(d-1)}} m. \quad (2.2.16)$$

One may check that the first law holds:

$$dM_{(d+1)} = T_{(d+1)}dS_{(d+1)} + \mu_{(d+1)}dQ_{(d+1)} \quad \Leftrightarrow \quad dG_{(d+1)} = -S_{(d+1)}dT_{(d+1)} - Q_{(d+1)}d\mu_{(d+1)} \quad (2.2.17)$$

and then examine global and local thermodynamical equilibrium in the grand-canonical ensemble by computing the Gibbs potential, the heat capacity and the electric permittivity⁴:

$$G_{(d+1)} = \frac{V_{(d-1)} \left(1 - \frac{d-1}{2}\delta_1^2\right) r_+^{d-\frac{d-1}{2}\delta_1^2}}{16\pi G_N^{(d-1)}} \left[-\ell^2 + \frac{(d-2)\lambda_{(d-1)}r_+^{(d-1)\delta_1^2-2}}{\left(1 - \frac{d-1}{2}\delta_1^2\right) \left(d - 2 + \frac{d-1}{2}\delta_1^2\right)} - \frac{q^2}{r_+^{(d-1)(2-(d-1)\delta_1^2)}} \right] \quad (2.2.18)$$

$$C_\mu^{(d+1)} = T \frac{dS}{dT} \Big|_\mu, \quad \epsilon_T^{(d+1)} = \frac{dQ}{d\mu} \Big|_T. \quad (2.2.19)$$

One finds out that when $\delta_1^2 \leq \delta_c^2$, the thermodynamics is identical to that of a charged AdS black hole, [41, 40].

Diagonal oxidation for $\gamma^2 \leq \delta_c^2$ In this case, the lower-dimensional gauge field (2.2.9) originates from a higher-dimensional Maxwell field strength in the action, as described above. Thus, from the result of section 2.1, we can expect to recover the Reissner-Nordström solution in $(2\sigma + 1)$ dimensions, using the Ansatz (2.1.2). This is indeed what

⁴Both are straightforward to compute, but the expressions are cumbersome.

happens and the solution (2.2.6) uplifts to:

$$ds_{(2\sigma+1)}^2 = -\frac{f(\rho)}{\ell_{(2\sigma+1)}^2} d\tau^2 + \frac{\ell_{(2\sigma+1)}^2 d\rho^2}{4\rho^2 f(\rho)} + \rho^{-1} \left(dX_{(d-1)}^2 + dX_{(2\sigma-d)}^2 \right), \quad (2.2.20)$$

$$f(\rho) = 1 + \ell_{(2\sigma+1)}^2 (\lambda_{(2\sigma-1)} \rho - m \rho^\sigma + q^2 \rho^{2\sigma-1}), \quad (2.2.21)$$

$$A = -\sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} d\tau. \quad (2.2.22)$$

To obtain this, we have used the same change of coordinates (2.1.25) as in section 2.1, as well as rescaled the charge parameter $q \rightarrow (2\sigma-1)q/(d-1)$.

The thermodynamic quantities become:

$$T_{(2\sigma+1)} = \frac{\rho_+^{-\frac{1}{2}}}{4\pi} \left[\frac{2\sigma}{\ell_{(2\sigma+1)}^2} + (2\sigma-2)\lambda_{(2\sigma-1)}\rho_+ - (2\sigma-2)q^2\rho_+^{(2\sigma-1)} \right], \quad (2.2.23)$$

$$S_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{4G_N^{(2\sigma+1)}} \rho_+^{-\frac{1}{2}(2\sigma-1)}, \quad (2.2.24)$$

$$Q_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{16\pi G_N^{(2\sigma+1)}} q \sqrt{2(2\sigma-1)(2\sigma-2)}, \quad (2.2.25)$$

$$\mu_{(2\sigma+1)} = \sqrt{2\frac{2\sigma-1}{2\sigma-2}} q \rho_+^{\sigma-1}, \quad (2.2.26)$$

$$M_{(2\sigma+1)} = \frac{V_{(2\sigma-1)}}{16\pi G_N^{(2\sigma+1)}} (2\sigma-1)m, \quad (2.2.27)$$

$$G_{(2\sigma+1)} = -\frac{V_{(2\sigma-1)}\rho_+^{-\sigma}}{16\pi G_N^{(2\sigma+1)}} \left[\ell_{(2\sigma+1)}^2 - \lambda_{(2\sigma-1)}\rho_+ + q^2\rho_+^{(2\sigma-1)} \right], \quad (2.2.28)$$

again, coinciding with results from [40, 41].

When $\delta_c^2 < \delta_1^2 < \delta_{max}^2$, one can oxidize the solution using (2.1.6) and $\gamma = \delta_2 = \frac{2}{(d-1)\delta_1}$ (which in particular means $\gamma^2 \leq \delta_c^2$). This leads to the same solution (2.2.22).

2.2.2 Solution with a single exponential potential

Let us now consider the case of the potential with a single exponential: $\Lambda_2 = 0$ and $\gamma = \delta_2 = \delta_c^2/\delta_1$ in (2.2.2). The field equations can be integrated to the following solution (see [42] for its four-dimensional version):

$$ds_{(d+1)}^2 = -e^{\gamma\phi} \frac{V(p)}{p^2} dt^2 + \frac{e^{\delta_1\phi} V(p)^{-1} dp^2}{(-\Lambda_1)(\delta_{max}^2 - \delta_1^2)} + e^{\delta_1\phi} (p - p_-)^2 \frac{(\delta^2 - \delta_1^2)}{(\delta_{max}^2 - \delta_1^2)} dR_{(d-1)}^2, \quad (2.2.29a)$$

$$V(p) = (p - p_+)(p - p_-), \quad (2.2.29b)$$

$$e^\phi = e^{\phi_0} p^{\frac{2\delta_1}{[(d-2)\delta_1^2 + \delta_c^2]}} (p - p_-)^{\frac{2(d-1)\delta_1(\delta_1^2 - \delta_c^2)}{[(d-2)\delta_1^2 + \delta_c^2][\delta_{max}^2 - \delta_1^2]}}, \quad (2.2.29c)$$

$$\mathcal{A} = \sqrt{\frac{2(d-1)\delta_1^2 p_-}{[(d-2)\delta_1^2 + \delta_c^2] p_+}} \left(1 - \frac{p_+}{p} \right) dt, \quad (2.2.29d)$$

$$\gamma\delta_1 = \delta_c^2, \quad \delta_c^2 = \frac{2}{d-1}, \quad \delta_{max}^2 = \frac{2d}{d-1}.$$

$p = 0$ and p_- are both curvature singularities, while p_+ is an event horizon. Setting the charge to zero, the neutral solution (2.1.9) is recovered with $\lambda_{(d-1)} \sim \Lambda_2 = 0$.

The thermodynamics of this solution in $d + 1 = 4$ were studied in [6]. These results generalize to higher dimensions straightforwardly. For the black hole solutions, that is when

$$\Lambda_1 < 0 \quad \&\& \quad \delta_1^2 < \delta_{max}^2 \quad (2.2.30)$$

holds, three ranges should be distinguished:

1. Lower range: $\delta_1^2 \leq \delta_c^2$. The solution behaves as the charged AdS planar black hole, there is a single locally stable branch, both in the canonical and grand-canonical ensembles.
2. Middle range: $\delta_c^2 < \delta_1^2 < \frac{d^2-3}{(2d-3)(d-1)} + \frac{\sqrt{d+3}}{2d-3}$. The solution behaves as the charged (asymptotically flat) Reissner-Nordström black hole, there are two branches, small and large black holes, only the latter of which are locally stable.
3. Upper range: $\frac{d^2-3}{(2d-3)(d-1)} + \frac{\sqrt{d+3}}{2d-3} \leq \delta_1^2 < \delta_{max}^2$. The solution behaves as the (asymptotically flat) Schwarzschild black hole and is always locally unstable.

Diagonal curved oxidation $\delta_c^2 < \delta_1^2 < \delta_{max}^2$. In this range of values of δ_1 , we expect that the uplift of (2.2.29) should give a charged, asymptotically flat version of the black brane (2.1.38). Indeed, using the diagonal Ansatz (2.1.6) together with (2.1.7) as in (2.2.5), Λ_1 plays the role of the curvature of the internal space instead of that of a higher-dimensional cosmological constant: this brings us to $(2\sigma + 1)$ -dimensional Einstein-Maxwell theory, *without cosmological constant*. After the following change of coordinates and identifications:

$$p = \rho^{2\sigma-d-1}, \quad p_{\pm} = \rho_{\pm}^{2\sigma-d-1}, \quad (2.2.31)$$

as well as

$$t = \sqrt{\lambda_{(2\sigma-d)}} \tau, \quad -2\Lambda = (2\sigma - d)(2\sigma - d - 1)\lambda_{(2\sigma-d)} \quad (2.2.32)$$

the $(2\sigma + 1)$ -dimensional solution is:

$$\begin{aligned} ds_{(2\sigma+1)}^2 = & -f(\rho)d\tau^2 + \left[1 - \left(\frac{\rho_-}{\rho}\right)^{2\sigma-d-1}\right]^{\frac{2(d-1)}{(2\sigma-2)(2\sigma-d-1)}} \left[\frac{d\rho^2}{f(\rho)} + \rho^2 dK_{(2\sigma-d)}^2\right] + \\ & + \left[1 - \left(\frac{\rho_-}{\rho}\right)^{2\sigma-d-1}\right]^{\frac{-2}{2\sigma-2}} dR_{(d-1)}^2, \end{aligned} \quad (2.2.33)$$

$$f(\rho) = \lambda_{(2\sigma-d)} \left[1 - \left(\frac{\rho_+}{\rho}\right)^{2\sigma-d-1}\right] \left[1 - \left(\frac{\rho_-}{\rho}\right)^{2\sigma-d-1}\right], \quad (2.2.34)$$

$$A = -\sqrt{\frac{2(2\sigma-1)}{(2\sigma-2)\lambda_{(2\sigma-d)}}} \left(\frac{\rho_-\rho_+}{\rho^2}\right)^{2\sigma-d-1} d\tau. \quad (2.2.35)$$

This solution can be better interpreted by going to

$$r^{2\sigma-d-1} = \rho^{2\sigma-d-1} - \rho_-^{2\sigma-d-1}, \quad (2.2.36)$$

$$\rho_-^{2\sigma-d-1} = r_0^{2\sigma-d-1} \frac{\sinh^2 \omega}{\lambda_{(2\sigma-d)}}, \quad \rho_+^{2\sigma-d-1} = r_0^{2\sigma-d-1} \frac{\cosh^2 \omega}{\lambda_{(2\sigma-d)}}, \quad (2.2.37)$$

we find

$$ds_{(2\sigma+1)}^2 = -K(r)^{-2} f(r) d\tau^2 + K(r)^{\frac{2}{2\sigma-2}} \left[\frac{dr^2}{f(r)} + r^2 dX_{(2\sigma-d)}^2 + dR_{(d-1)}^2 \right], \quad (2.2.38)$$

$$f(r) = \lambda_{(2\sigma-d)} - \left(\frac{r_0}{r} \right)^{2\sigma-d-1}, \quad K(r) = 1 + \frac{\sinh^2 \omega}{\lambda_{(2\sigma-d)}} \left(\frac{r_0}{r} \right)^{2\sigma-d-1}, \quad (2.2.39)$$

$$A = -\sqrt{\frac{2(2\sigma-1)\lambda_{(2\sigma-d)}}{(2\sigma-2)}} (1 - K(r)^{-1}) \coth \omega d\tau. \quad (2.2.40)$$

This is a $(d-1)$ -brane supporting a point-like electric charge, [43]. It can be obtained from the $(d-1)$ -brane with a q -charge (corresponding to a $(q+1)$ -form potential in the theory), where only $q \leq d-1$ directions of the worldvolume of the brane support the charge, [44]. Taking $q = 0$ recovers the solution (2.2.38). It can also be obtained by uplifting the asymptotically flat dilatonic black holes of [45]. This points out an interesting relation between asymptotically flat black holes and black branes with an exponential potential, since they are mapped to each other by Kaluza-Klein oxidation/reduction, depending on whether one reduces on the worldvolume of the brane or on the compact space $\mathbf{X}_{(2\sigma-d)}$.

2.3 Non-diagonal reduction to Einstein-Maxwell-Dilaton theories

In this section, we consider adding a Maxwell gauge field to the action (2.1.4), as in (2.2.2), this time generated in the reduction by turning on a Kaluza-Klein vector.

The metric Ansatz has an off-diagonal component along one of the reduced directions. This reduction is consistent (when only one scalar field is kept) only when the reduction is along an \mathbf{S}^1 , see Appendix A.2:

$$ds_{(d+2)}^2 = e^{-\delta_1 \phi} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta_1} (\delta_c^2 - \delta_1^2)} (dy + \mathcal{A})^2, \quad \mathcal{A} = \mathcal{A}_M dx^M. \quad (2.3.1)$$

As the reduction is only over a single dimension, one has to set

$$2\sigma = d+1 \quad \Rightarrow \quad R_{(1)} = 0 \quad (2.3.2)$$

since the \mathbf{S}^1 has zero curvature. Then, the action (2.2.2) is recovered with

$$\Lambda_2 = 0, \quad \delta_1 = \sqrt{\frac{2}{d(d-1)}} < \delta_c, \quad \gamma = \delta_2 = \frac{\delta_c^2}{\delta_1} = \sqrt{\frac{2d}{d-1}} > \delta_c. \quad (2.3.3)$$

Alternatively, one may reduce with

$$ds_{(d+2)}^2 = e^{-\frac{\delta_c^2}{\delta_1} \phi} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta_1} (\delta_1^2 - \delta_c^2)} (dy + \mathcal{A})^2, \quad \mathcal{A} = \mathcal{A}_M dx^M, \quad (2.3.4)$$

and recover (2.2.2) with

$$\Lambda_1 = 0, \quad \delta_1 = \sqrt{\frac{2d}{(d-1)}} > \delta_c, \quad \gamma = \delta_1 = \sqrt{\frac{2d}{d-1}} > \delta_c. \quad (2.3.5)$$

Both reduction schemes yield $\gamma > \delta_c$, and thus are complementary to those presented in section 2.2. The price to pay is that there is a single exponential in the potential, and that the parameters δ_1 , γ have a fixed value. We will discuss a more general reduction that allows for generic σ in section 2.4.

We shall now consider the uplifts of the previous two charged solutions (2.2.6) (restricted to a single exponential potential) and (2.2.29a).

Non-diagonal oxidation to $(d+2)$ dimensions: $\delta_1^2 \geq \delta_c^2$. Let us take the single-exponential restriction $\Lambda_1 = 0$ of the solution (2.2.6) and use the Ansatz (2.3.4) as well as (2.3.5). In that case, we expect to recover the Schwarzschild-AdS black brane (2.1.38) carrying a wave, since now Λ_2 is identified with the higher-dimensional cosmological constant. Let us call r_{\pm} the roots of the black-hole potential $V(r)$ (2.2.7), which we can rewrite

$$V(r) = -\lambda_{(d-1)} \frac{(d-2)}{2(d-1)^2} r^{2d} \left(1 - \left(\frac{r_+}{r}\right)^{2d-2}\right) \left(1 - \left(\frac{r_-}{r}\right)^{2d-2}\right), \quad (2.3.6)$$

$$A = \sqrt{\frac{(-\lambda_{(d-1)})(d-2)}{2(d-1)^2}} \left(\frac{r_-}{r_+}\right)^{d-1} \left(1 - \left(\frac{r_+}{r}\right)^{2d-2}\right) dt \quad (2.3.7)$$

where one has to keep in mind that, since $\gamma^2 > \delta_c^2$, $\lambda_{(d-1)} < 0$ and hence there is an overall minus sign in $V(r)$.

After some manipulations, the higher-dimensional metric is:

$$\begin{aligned} ds_{(d+2)}^2 = & -r^{2d-2} \left(1 - \left(\frac{r_-}{r}\right)^{2d-2}\right) \left[\sqrt{\frac{(-\lambda_{(d-1)})(d-2)m}{2(d-1)^2 r_+^{2d-2}}} dt - \sqrt{\frac{r_-^{2d-2}}{m}} dy \right]^2 + \\ & + \frac{2(d-1)^2 r^{-2} dr^2}{(-\lambda_{(d-1)})(d-2) \left(1 - \left(\frac{r_+}{r}\right)^{2d-2}\right) \left(1 - \left(\frac{r_-}{r}\right)^{2d-2}\right)} + \\ & + \frac{r_+^{2d-2}}{m} r^{2d-2} \left(1 - \left(\frac{r_-}{r}\right)^{2d-2}\right) dy^2 + dX_{(d-1)}^2, \end{aligned} \quad (2.3.8)$$

where we have set $m = r_+^{2d-2} - r_-^{2d-2}$ and $\mathbf{X}_{(d-1)}$ is a negative curvature space, $\lambda_{(d-1)} < 0$.

We can bring this last expression to a more standard form by changing to the coordinate

$$\rho^{-1} = r^{2d-2} - r_-^{2d-2}, \quad r_+^{2d-2} = m \cosh^2 \omega, \quad r_-^{2d-2} = m \sinh^2 \omega, \quad (2.3.9)$$

and replacing

$$d(d-2)\lambda_{(d-1)} = 2\Lambda_2 = 2\Lambda. \quad (2.3.10)$$

Then:

$$\begin{aligned} ds_{(d+2)}^2 = & -\rho^{(-1)} (1 - m\rho) \left[\frac{dt}{\cosh \omega} - \sinh \omega dy \right]^2 + \frac{\cosh^2 \omega}{\rho} dy^2 + \\ & + \frac{d\rho^2}{(-4\Lambda)\rho^2 (1 - m\rho)} + dX_{(d-1)}^2. \end{aligned} \quad (2.3.11)$$

This is just the boosted three-dimensional B(H)TZ black hole, [46, 47], times a hyperbolic plane $\mathbf{X}_{(d-1)}$, whose curvature is fixed by (2.3.10).

Non-diagonal oxidation to $d + 2$ dimensions: $\delta_1^2 \leq \delta_c^2$. Let us uplift the solution (2.2.29) with the Ansatz (2.3.1) and use (2.3.3):

$$ds_{(d+2)}^2 = \frac{ddp^2}{(-2\Lambda)(d+1)V(p)} + (p - p_-)^{\frac{2}{d+1}} dR_{(d-1)}^2 + \\ + p(p - p_-)^{-\frac{d+1}{d-1}} \left[\left(dy + \sqrt{\frac{p_-}{p_+}} \frac{p - p_+}{p} dt \right)^2 - \frac{V(p)}{p^2} dt^2 \right]. \quad (2.3.12)$$

Changing coordinates,

$$p - p_- = r^{d+1}, \quad p_+ = m \cosh^2 \omega, \quad p_- = m \sinh^2 \omega, \quad (2.3.13)$$

leads to a more more standard form of the metric:

$$ds_{(d+2)}^2 = -\frac{r^2 f(r)}{K(r)} dt^2 + \frac{\ell_{(d+2)}^2 dr^2}{f(r)} + r^2 dR_{(d-1)}^2 + r^2 K(r) \left(dy + \tanh \omega \frac{f(r)}{K(r)} dt \right)^2 \quad (2.3.14)$$

$$f(r) = r^2 - \frac{m}{r^{d-1}}, \quad K(r) = 1 + \frac{m \sinh^2 \omega}{r^{d+1}}. \quad (2.3.15)$$

A further change of the radial coordinate, $r^2 = 1/\rho$, allows to recover the form of the metric (3.3.3) used in section 3.3. After some manipulation of the dy^2 terms and rescaling $\bar{y} = \cosh \omega y$, the uplifted metric can be rewritten as:

$$ds_{(d+2)}^2 = -\frac{f(\rho)}{\rho} \left(\frac{dt}{\cosh \omega} - \tanh \omega d\bar{y} \right)^2 + \frac{\ell_{(d+2)}^2 d\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho} \left(dR_{(d-1)}^2 + d\bar{y}^2 \right), \quad (2.3.16)$$

$$f(\rho) = 1 - m\rho^{\frac{1}{2}(d+1)}, \quad (2.3.17)$$

which corresponds to Schwarzschild-AdS $_{d+2}$ carrying a wave. The cylindrical black string in four dimensions and its related stationary version have been studied in [48, 49]. The generalisation of the stationary cylindrical black hole to $d + 1$ dimensions and to $[d/2]^5$ arbitrary rotation parameters was presented in [50]. Making the change of coordinates:

$$t = \cosh \omega \bar{t} + \sinh \omega \bar{y}, \quad (2.3.18)$$

shows that the metric (2.3.17) is locally isometric the static black brane. This is only a local isometry because we are mixing one periodic coordinate (y) with the time coordinate. It is reflected in the fact that the first Betti number of this spacetime is not zero, and all closed curves are not in the same equivalence class, [51] (the ones wrapped around the cylinder cannot be shrunk to a point).

If one unwraps the extra coordinate and takes its universal covering, then this stationary spacetime becomes globally isometric to the static AdS black brane, by boosting it along the worldvolume direction y . Now, a boost would usually mean the following coordinate transformation:

$$t = \cosh \omega \bar{t} + \sinh \omega \bar{y}, \quad y = \sinh \omega \bar{t} + \cosh \omega \bar{y}. \quad (2.3.19)$$

⁵ $\lceil \rceil$ means the integer part

One can show, reversing the previous steps, that this change of coordinates in the AdS black brane metric gives back, after reduction along the boost direction, the solution (2.2.29) but where the gauge field has been shifted so that it has zero chemical potential at spatial infinity.

Finally, boosted black branes in the context of the AdS/CFT correspondence were investigated in [52]. The thermodynamics for both this spacetime and its lower-dimensional reduction can be recovered from the formulæ in section 3.3, by setting $2\sigma = d + 1$ and subsequently turning off the extra scalar ζ .

2.4 Generalized non-diagonal reduction along a torus

The non-diagonal reduction discussed in the previous subsection was restricted to the reduction over a single \mathbf{S}^1 . We would like to generalize the reduction to a torus reduction and then consider a continuation over its dimension. The generic, non-diagonal reduction of Einstein theory with a 4-form field strength was performed in [53] for $D = 11$ supergravity. It is straightforward to generalize the formulæ to $2\sigma + 1$ dimensions with a cosmological constant and we present these results in this section. As in the two previous subsections, straight latin capital fields refer to higher-dimensional gauge fields, while calligraphic letters are reserved for lower-dimensional gauge fields stemming from the reduction.

Our starting point is the AdS-Maxwell action in (2.0.1) and we would like to make a general non-diagonal torus reduction over $\mathbf{T}^{(2\sigma-d)}$. The lower-dimensional fields are the metric, $(2\sigma-d)$ scalar fields $\vec{\phi}$ parametrizing the size of the torus, gauge fields, $\mathcal{A}_{(1)}^a = \mathcal{A}_{(1)M}^a dx^M$ and $A_{(1)} = A_{(1)M} dx^M$, originating from the metric and higher-dimensional gauge field and axions $\mathcal{A}_{(0)b}^a$ ($a < b \leq 2\sigma - d$) and $A_{(0)a}$ ($a \leq 2\sigma - d$). The reduction ansatz is⁶

$$ds_{(2\sigma+1)}^2 = e^{-\vec{\delta} \cdot \vec{\phi}} ds_{(d+1)}^2 + \sum_{a=d+1}^{2\sigma} e^{-\vec{\gamma}_a \cdot \vec{\phi}} (h^a)^2, \quad (2.4.1)$$

$$h^a = dy^a + \mathcal{A}_{(1)}^a + \sum_{b=a+1}^{2\sigma} \mathcal{A}_{(0)b}^a dy^b, \quad (2.4.2)$$

$$A_{(1)}^{(2\sigma+1)} = A_{(1)M}^{(d+1)} dx^M + A_{(0)a}^{(d+1)} dy^a, \quad F_{(2)}^{(2\sigma+1)} = \tilde{F}_{(2)}^{(d+1)} + \tilde{F}_{(1)a}^{(d+1)} h^a, \quad (2.4.3)$$

where we have explicitly made the distinction between higher- and lower-dimensional gauge fields. We define the tilded fields just below. This leads to a reduced theory governed by the action

$$S_{(d+1)} = \frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2} (\partial \vec{\phi})^2 - 2\Lambda e^{-\vec{\delta} \cdot \vec{\phi}} - \frac{1}{4} \sum_a e^{-\vec{\beta}_a \cdot \vec{\phi}} (\tilde{\mathcal{F}}_{(2)}^a)^2 \right. \\ \left. - \frac{1}{2} \sum_{a < b} e^{-\vec{\beta}_{ab} \cdot \vec{\phi}} (\tilde{\mathcal{F}}_{(1)}^{ab})^2 - \frac{1}{4} e^{\vec{\delta} \cdot \vec{\phi}} (\tilde{F}_{(2)})^2 - \frac{1}{2} \sum_a e^{-\vec{\alpha}_a \cdot \vec{\phi}} (\tilde{F}_{(1)a})^2 \right], \quad (2.4.4)$$

⁶The massive modes of the KK tower transform as doublets of the isometry group, while the massless modes transform as singlets. Since the isometry group is Abelian, the two representations do not mix. The massive modes are then not sourced by the massless modes, and can be safely truncated, [54].

$$\tilde{\mathcal{F}}_{(2)}^a = \mathcal{F}_{(2)}^a - \gamma^b{}_c \mathcal{F}_{(1)b}^a \wedge \mathcal{A}_{(1)}^c, \quad \mathcal{F}_{(2)}^a = d\mathcal{A}_{(1)}^a \quad (2.4.5)$$

$$\tilde{\mathcal{F}}_{(1)b}^a = \gamma^c{}_b \mathcal{F}_{(1)c}^a, \quad \mathcal{F}_{(1)b}^a = d\mathcal{A}_{(0)b}^a, \quad (2.4.6)$$

$$\tilde{F}_{(2)} = F_{(2)} - \tilde{F}_{(1)a} \wedge \mathcal{A}_{(1)}^a, \quad F_{(2)} = dA_{(1)} \quad (2.4.7)$$

$$\tilde{F}_{(1)a} = \gamma^b{}_a F_{(1)b}, \quad F_{(1)} = dA_{(0)}, \quad (2.4.8)$$

$$\gamma^a{}_b = [(1 + \mathcal{A}_{(0)})^{-1}]^a{}_b = \delta^a{}_b - \mathcal{A}_{(0)b}^a + \mathcal{A}_{(0)b}^c \mathcal{A}_{(0)c}^a + \dots \quad (2.4.9)$$

where the tilded field strengths include extra transgression terms⁷ compared to their untilded, usual definition as the exterior derivative of the gauge potential. The axionic metric $\gamma^a{}_b$ has a finite number of terms since a particular axion $\mathcal{A}_{(0)b}^a$ is only defined for $a < b$ and $\vec{\delta}, \vec{\gamma}_a, \vec{\beta}_a, \vec{\alpha}_a, \vec{\beta}_{ab}$ are given by:

$$\vec{\delta} = (\delta_{d+1}, \dots, \delta_{2\sigma}), \quad \delta_a = \sqrt{2/((2\sigma - a)(2\sigma - a - 1))} \quad (2.4.10)$$

$$\vec{f}_a = \left(\underbrace{0, \dots, 0}_{a-1}, (2\sigma - a)\delta_{d+a}, \delta_{d+1+a}, \dots, \delta_{2\sigma} \right), \quad (2.4.11)$$

$$\vec{\gamma}_a = \vec{\delta} - \vec{f}_a = \left(\delta_{d+1}, \dots, -(2\sigma - a - 1)\delta_{d+a}, \underbrace{0, \dots, 0}_{2\sigma - d - a} \right), \quad (2.4.12)$$

$$\vec{\alpha}_a = \vec{f}_a - \vec{\delta}, \quad \vec{\beta}_a = -\vec{f}_a, \quad \vec{\beta}_{ab} = -\vec{f}_a + \vec{f}_b. \quad (2.4.13)$$

Let us now work out how we may generate solutions to the equations of motion stemming from (2.4.4). We may start from the charged planar AdS black hole and boost it along $2\sigma - d$ directions of the horizon, [50], with ω_a the boost parameters:

$$\begin{aligned} ds_{(2\sigma+1)}^2 = & -\frac{f(\rho)}{\rho} \left(\xi d\tau - \sum_{a=1}^{2\sigma-d} \omega_a dy^a \right)^2 + \frac{\rho^{-1}}{\ell_{(2\sigma+1)}^4} \sum_{a=1}^{2\sigma-d} \left(\omega_a d\tau - \xi \ell_{(2\sigma+1)}^2 dy^a \right)^2 - \\ & -\frac{\rho^{-1}}{\ell_{(2\sigma+1)}^2} \sum_{a < b} \left(\omega_a dy^b - \omega_b dy^a \right)^2 + \rho^{-1} dR_{(d-1)}^2 + \frac{d\rho^2}{4\rho^2 f(\rho)}, \end{aligned} \quad (2.4.14)$$

$$f(\rho) = \ell_{(2\sigma+1)}^{-2} - m\rho^\sigma + q^2 \rho^{2\sigma-1}, \quad (2.4.15)$$

$$A = -\sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} \left(\xi d\tau - \sum_{a=1}^{2\sigma-d} \omega_a dy^a \right), \quad (2.4.16)$$

$$\xi = 1 + \sum_{a=1}^{2\sigma-d} \frac{\omega_a^2}{\ell_{(2\sigma+1)}^2}. \quad (2.4.17)$$

It is now a matter of calculation to show that all the terms in the (τ, y^a) sector can be rearranged as

$$ds_{(2\sigma+1)}^2 = -\frac{f(\rho)}{\rho K_{2\sigma-d}} d\tau^2 + \rho^{-1} dR_{(d-1)}^2 + \frac{d\rho^2}{4\rho^2 f(\rho)} + \sum_a \frac{K_a}{\rho K_{a-1}} (h^a)^2, \quad (2.4.18)$$

⁷It is the tilded field strengths which compare directly with their higher-dimensional counterparts, as can be seen from (2.4.3).

$$h^a = dy^a - \frac{\omega_a}{\sum_{b \leq a} \omega_b^2} (1 - K_a^{-1}) \left(\xi d\tau - \sum_{b > a} \omega_b dy^b \right), \quad (2.4.19)$$

$$K_a(\rho) = 1 + \sum_{b \leq a} \omega_b^2 (m\rho^\sigma - q^2 \rho^{2\sigma-1}), \quad (2.4.20)$$

where by convention $\omega_0 = 0$ and thus $K_0 = 1$. This is precisely the form we need to match the Kaluza-Klein reduction Ansatz (2.4.1). Thus, one can show that the $(d+1)$ -dimensional theory (2.4.4) admits the following solution

$$ds_{(d+1)}^2 = -\frac{\rho^{-\frac{2\sigma-1}{d-1}} f(\rho)}{(K_{2\sigma-d})^{\frac{d-2}{d-1}}} d\tau^2 + \rho^{-\frac{2\sigma-d}{d-1}} \frac{(K_{2\sigma-d})^{\frac{1}{d-1}}}{4\rho^2 f(\rho)} d\rho^2 + \rho^{-\frac{2\sigma-1}{d-1}} dR_{(d-1)}^2, \quad (2.4.21)$$

$$e^{\phi_a} = \rho^{-\frac{1}{2}(2\sigma-1)\delta_i} (K_a)^{\frac{1}{(2\sigma-a-1)\delta_a}} (K_{a-1})^{\frac{-1}{(2\sigma-a)\delta_a}}, \quad (2.4.22)$$

$$\mathcal{A}_{(1)}^a = -\frac{\omega_a}{\sum_{b \leq a} \omega_b^2} (1 - (K_a)^{-1}) \xi d\tau, \quad (2.4.23)$$

$$\mathcal{A}_{(0)b}^a = \frac{\omega_a \omega_b}{\sum_{b \leq a} \omega_b^2} (1 - (K_a)^{-1}), \quad (2.4.24)$$

$$A_{(1)} = -\sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} \xi d\tau, \quad (2.4.25)$$

$$A_{(0)a} = \sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} \omega_a. \quad (2.4.26)$$

If one wishes to delete any kind of reference to the higher-dimensional theory from which this solution originates, it suffices to replace $2\sigma = d + \mathcal{N}$, where \mathcal{N} is now simply the number of calligraphic gauge fields. Let us stress at this point that the procedure by which we obtained the above solution is not quite trivial, since there was no guarantee that we could reach the Kaluza-Klein form from the boosted black brane.

We may not analytically continue \mathcal{N} to arbitrary real values yet, since we cannot analytically continue the number of gauge fields! One may remedy this by using a mixed diagonal/non-diagonal reduction: first, reduce diagonally along $\mathcal{N} - \mathcal{M}$ dimensions, then non-diagonally along \mathcal{M} dimensions. One can then continue analytically the number of diagonal dimensions, *i.e.* $\mathcal{N} - \mathcal{M}$. In practice, setting

$$\forall a, b \leq \mathcal{N} - \mathcal{M} \quad \mathcal{A}_{(1)}^a = \mathcal{A}_{(0)b}^a = 0 \quad \Rightarrow \quad h^a = dy^a, \quad (2.4.27)$$

as well as

$$\frac{d + \mathcal{M} - 1}{2} \delta^2 = \frac{\mathcal{N} - \mathcal{M}}{d + \mathcal{N} - 1} \Leftrightarrow \mathcal{N} = \frac{\mathcal{M} + (d-1)(d + \mathcal{M} - 1)\frac{\delta^2}{2}}{1 - (d + \mathcal{M} - 1)\frac{\delta^2}{2}}, \quad (2.4.28)$$

and

$$e^{\frac{\Phi}{\delta}(\frac{2}{d+\mathcal{M}-1}-\delta^2)} = e^{-\vec{\gamma}_a \cdot \vec{\phi}} \quad \forall a \leq \mathcal{N} - \mathcal{M} \quad (2.4.29)$$

gives

$$\sum_{a=1}^{\mathcal{N}-\mathcal{M}} \phi_a^2 = \Phi^2. \quad (2.4.30)$$

The reduction Ansatz now is

$$ds_{(2\sigma+1)}^2 = e^{-\vec{\delta} \cdot \vec{\phi} - \delta \Phi} ds_{(d+1)}^2 + e^{-\delta \Phi} dR_{(\mathcal{N}-\mathcal{M})}^2 + \sum_a^{\mathcal{M}} e^{-\vec{\gamma}_a \cdot \vec{\phi}} (h^a)^2, \quad (2.4.31)$$

$$\delta_a = \sqrt{\frac{2}{(\mathcal{M} + d - a)(\mathcal{M} + d - a - 1)}} \quad a = 1 \dots \mathcal{M}, \quad (2.4.32)$$

$$\vec{f}_a = \left(\underbrace{0, \dots, 0}_{a-1}, (\mathcal{M} + d - a)\delta_{d+a}, \delta_{d+1+a}, \dots, \delta_{\mathcal{M}+d} \right), \quad (2.4.33)$$

$$\vec{\gamma}_a = \vec{\delta} - \vec{f}_a = \left(\delta_{d+1}, \dots, -(\mathcal{M} + d - a - 1)\delta_{d+a}, \underbrace{0, \dots, 0}_{\mathcal{M}-a} \right), \quad (2.4.34)$$

where the diagonal reduction runs over $\mathcal{N} - \mathcal{M}$ dimensions and the non-diagonal one over \mathcal{M} and the various (arrowed) vectors are \mathcal{M} -dimensional. The $(d+1)$ -dimensional action is

$$\begin{aligned} S_{(d+1)} = & \frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2} \partial \Phi^2 - \frac{1}{2} \left(\partial \vec{\phi} \right)^2 - 2\Lambda e^{-\vec{\delta} \cdot \vec{\phi} - \delta \Phi} \right. \\ & - \frac{1}{4} \sum_a^{\mathcal{M}} e^{-\vec{\beta}_a \cdot \vec{\phi}} \left(\tilde{\mathcal{F}}_{(2)}^a \right)^2 - \frac{1}{2} \sum_{a < b}^{\mathcal{M}} e^{-\vec{\beta}_{ab} \cdot \vec{\phi}} \left(\tilde{\mathcal{F}}_{(1)}^{ab} \right)^2 \\ & \left. - \frac{1}{4} e^{\vec{\delta} \cdot \vec{\phi} + \delta \Phi} \left(\tilde{F}_{(2)} \right)^2 - \frac{1}{2} \sum_a^{\mathcal{M}} e^{\delta \Phi - \vec{\alpha}_a \cdot \vec{\phi}} \left(\tilde{F}_{(1)a} \right)^2 \right], \end{aligned} \quad (2.4.35)$$

which has a solution

$$ds_{(d+1)}^2 = -\frac{\rho^{-\frac{2\sigma-1}{d-1}} f(\rho)}{(K_{2\sigma-d})^{\frac{d-2}{d-1}}} d\tau^2 + \rho^{-\frac{2\sigma-d}{d-1}} \frac{(K_{2\sigma-d})^{\frac{1}{d-1}}}{4\rho^2 f(\rho)} d\rho^2 + \rho^{-\frac{2\sigma-1}{d-1}} dR_{(d-1)}^2, \quad (2.4.36)$$

$$e^\Phi = \rho^{-\frac{1}{2} \frac{(d+\mathcal{M}-1)\delta}{1-\frac{d+\mathcal{M}-1}{2}\delta^2}}, \quad (2.4.37)$$

$$2\sigma = \frac{\mathcal{M} + d - (d + \mathcal{M} - 1)\frac{\delta^2}{2}}{1 - (d + \mathcal{M} - 1)\frac{\delta^2}{2}}, \quad (2.4.38)$$

$$e^{\phi_a} = \rho^{-\frac{1}{2}(2\sigma-1)\delta_a} (K_a)^{\frac{1}{(2\sigma-a-1)\delta_a}} (K_{a-1})^{\frac{-1}{(2\sigma-a)\delta_a}}, \quad a = 1 \dots \mathcal{M} \quad (2.4.39)$$

$$\mathcal{A}_{(1)}^a = -\frac{\omega_a}{\sum_{b \leq a} \omega_b^2} \left(1 - (K_a)^{-1} \right) \xi d\tau, \quad a = 1 \dots \mathcal{M}, \quad (2.4.40)$$

$$\mathcal{A}_{(0)b}^a = \frac{\omega_a \omega_b}{\sum_{b \leq a} \omega_b^2} \left(1 - (K_a)^{-1} \right), \quad a, b = 1 \dots \mathcal{M}, \quad (2.4.41)$$

$$A_{(1)} = -\sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} \xi d\tau, \quad (2.4.42)$$

$$A_{(0)aa} = \sqrt{\frac{2(2\sigma-1)}{2\sigma-2}} q \rho^{\sigma-1} \omega_a, \quad a = 1 \dots \mathcal{M}. \quad (2.4.43)$$

Since δ is now taken to be any real number, so is 2σ , and we have thus generated a *family* of solutions depending on the real parameter δ . In the following, we shall restrict the simplest

case $\mathcal{M} = 1$, $q = 0$, with only one gauge field generated from the metric, two Kaluza-Klein scalars and no axion or higher-dimensional Maxwell field.

3. Holography from generalized dimensional reduction

We will now consider the simplest non-trivial case and work it out completely. This is the $\mathcal{M} = 1$, $q = 0$ case discussed in the previous section (one gauge field, two Kaluza-Klein scalars and no axion or higher-dimensional Maxwell field). We start by deriving the generalized Kaluza-Klein reduction map in the next subsection. Although the map has already been given in the previous section, the discussion in this section serves as an illustration of the steps involved in its derivation. Furthermore, we will also be able to connect more directly with the discussion in [25]. Then we will move on and use these results to derive the holographic dictionary, followed by the derivation of the universal holographic hydrodynamics.

3.1 Generalized dimensional reduction

We start from Einstein gravity with negative cosmological constant in $(2\sigma + 1)$ dimensions and consider a reduction that involves a Kaluza-Klein gauge field,

$$S_{(2\sigma+1)} = L_{AdS} \int d^{2\sigma+1}x \sqrt{-g_{(2\sigma+1)}} [R + 2\sigma(2\sigma - 1)]. \quad (3.1.1)$$

where $L_{AdS} = \ell_{(2\sigma+1)}^{2\sigma-1}/(16\pi G_{2\sigma+1})$, $\ell_{(2\sigma+1)}$ is the AdS radius and we used an appropriate Weyl rescaling to move $\ell_{(2\sigma+1)}$ as an overall constant in the action.

We use the following reduction ansatz for the theory on the torus $\mathbf{T}^{(2\sigma-d)}$

$$ds_{(2\sigma+1)}^2 = ds_{(d+1)}^2(\rho, z) + e^{2\phi_1(\rho, z)} (dy - A_M dx^M)^2 + e^{\frac{2\phi_2(\rho, z)}{(2\sigma-d-1)}} dy^a dy^a, \quad (3.1.2)$$

where $a = 1, \dots, (2\sigma - d - 1)$. The coordinates (y, y^a) are periodically identified with period $2\pi R$ and $x^M = (\rho, z^i)$ with $M = 0, \dots, d$. This is a consistent truncation, since the resulting lower-dimensional field equations are equivalent to the higher-dimensional field equations. The resulting lower-dimensional theory is governed by the action

$$S_{(d+1)} = L \int d^{d+1}x \sqrt{-g_{(d+1)}} e^{\phi_1 + \phi_2} \left[R + 2\partial\phi_1\partial\phi_2 + \frac{2\sigma - d - 2}{2\sigma - d - 1} (\partial\phi_2)^2 - \frac{1}{4} e^{2\phi_1} F_{MN} F^{MN} + 2\sigma(2\sigma - 1) \right]. \quad (3.1.3)$$

where $L = L_{AdS}(2\pi R)^{2\sigma-d}$.

One can derive this action as follows. First reduce on the $(2\sigma - d - 1)$ -dimensional torus to obtain

$$S_{(d+2)} = L_{AdS}(2\pi R)^{2\sigma-d-1} \int d^{d+2}x \sqrt{-g_{(d+2)}} e^{\phi_2} \left[R_{(d+2)} + \frac{2\sigma - d - 2}{2\sigma - d - 1} (\partial\phi_2)^2 + 2\sigma(2\sigma - 1) \right]. \quad (3.1.4)$$

To obtain this result we use the fact that

$$R_{(2\sigma+1)} = R_{(d+2)} - 2\nabla^2\phi_2 - \frac{2\sigma-d}{2\sigma-d-1}(\partial\phi_2)^2. \quad (3.1.5)$$

Now we reduce on the y direction including a Kaluza-Klein gauge field. Note that

$$ds_{(d+2)}^2 = ds_{(d+1)}^2 + e^{2\phi_1} (dy - A_M dx^M)^2, \quad (3.1.6)$$

and thus

$$R_{(d+2)} = R_{(d+1)} - 2\nabla^2\phi_1 - 2(\partial\phi_1)^2 - \frac{1}{4}e^{2\phi_1}F_{MN}F^{MN}. \quad (3.1.7)$$

Substituting into (3.1.4) leads to (3.1.3). Setting $F_{MN} = 0$, and rescaling

$$\phi_1 = \frac{\phi}{(2\sigma-d)}; \quad \phi_2 = \frac{\phi(2\sigma-d-1)}{(2\sigma-d)}, \quad (3.1.8)$$

with $\phi = (\phi_1 + \phi_2)$ results in the action for non-conformal branes derived in [24, 22],

$$S = L \int d^{d+1}x \sqrt{-g_{d+1}} e^\phi \left(R + \frac{2\sigma-d-1}{2\sigma-d} (\partial\phi)^2 + 2\sigma(2\sigma-1) \right). \quad (3.1.9)$$

It is natural to rewrite the action (3.1.3) in terms of the scalar

$$\psi = \phi_1 + \phi_2, \quad (3.1.10)$$

since the determinant of the metric over the torus is expressed in terms of ψ as $\sqrt{g_{T^{2\sigma-d}}} = e^\psi$. We also use the combination

$$\zeta = (2\sigma-d-1)\phi_1 - \phi_2, \quad (3.1.11)$$

in terms of which the reduction of the metric is

$$ds_{(2\sigma+1)}^2 = ds_{(d+1)}^2 + e^{2\frac{(\psi+\zeta)}{(2\sigma-d)}} (dy - A_M dx^M)^2 + e^{\frac{2\psi}{(2\sigma-d)} - \frac{2\zeta}{(2\sigma-d)(2\sigma-d-1)}} dy^a dy_a, \quad (3.1.12)$$

and the action becomes

$$S_{(d+1)} = L \int d^{d+1}x \sqrt{-g_{(d+1)}} e^\psi \left[R - \frac{1}{(2\sigma-d)(2\sigma-d-1)} (\partial\zeta)^2 + \frac{2\sigma-d-1}{2\sigma-d} (\partial\psi)^2 - \frac{1}{4} e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}} F_{MN} F^{MN} + 2\sigma(2\sigma-1) \right]. \quad (3.1.13)$$

Note that the equation of motion for ζ is

$$\nabla[e^\psi \partial\zeta] = \frac{1}{4} (2\sigma-d-1) e^\psi e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}} F_{MN} F^{MN}. \quad (3.1.14)$$

This implies that is always consistent to set $\zeta = 0$ when $F_{MN} = 0$: the action with both ζ and F set to zero is precisely that given above, with the identification $\psi = \phi$.

The equation of motion for ψ is

$$\nabla[e^\psi \partial\psi] = \frac{2\sigma-d}{2(2\sigma-d-1)} e^\psi \left[R - \frac{1}{(2\sigma-d)(2\sigma-d-1)} (\partial\zeta)^2 + \frac{2\sigma-d-1}{2\sigma-d} (\partial\psi)^2 - \frac{1}{4} \frac{(2\sigma-d+2)}{(2\sigma-d)} e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}} F_{MN} F^{MN} + 2\sigma(2\sigma-1) \right], \quad (3.1.15)$$

and the gravitational field equation is

$$\begin{aligned}
& R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{(2\sigma-d)(2\sigma-d-1)} \left(-\frac{1}{2}g_{MN}(\partial\zeta)^2 + \partial_M\zeta\partial_N\zeta \right) \\
& + \frac{1}{2\sigma-d} \left(\frac{1}{2}(2\sigma-d+1)g_{MN}(\partial\psi)^2 - \partial_M\psi\partial_N\psi \right) \\
& - \frac{1}{4}e^{\frac{2(\zeta+\psi)}{2\sigma-d}} \left(-\frac{1}{2}g_{MN}F_{PQ}F^{PQ} + 2F_M^Q F_{NQ} \right) - \sigma(2\sigma-1)g_{MN} \\
& - \nabla_N\nabla_M\psi + g_{MN}\square\psi = 0.
\end{aligned} \tag{3.1.16}$$

The gauge field equation is

$$\nabla_M \left(e^{\frac{2(\zeta+\psi)}{(2\sigma-d)}} F^{MN} \right) = 0. \tag{3.1.17}$$

From the equations of motion, one can notice that there are certain special values of σ . In the case of $2\sigma = (d+1)$, the reduction is along a circle and there is no additional scalar field ζ . The case of $2\sigma = d$ corresponds to the conformal case where there is no reduction at all, and one retains only the metric. One should note that it is also clearly consistent to set the gauge field to zero whilst retaining both scalars (ζ, ψ) . These field equations are, as mentioned above, completely equivalent to the higher-dimensional Einstein equations so the reduction is consistent.

We may conformally rescale the action to bring it into Einstein frame, using the following rescaling of the metric,

$$g_{MN} = e^{-2\psi/(d-1)} \bar{g}_{MN} \tag{3.1.18}$$

to get:

$$\begin{aligned}
S_{(d+1)} = L \int d^{d+1}x \sqrt{-\bar{g}_{(d+1)}} \left[\bar{R} - \frac{1}{(2\sigma-d)(2\sigma-d-1)} (\partial\zeta)^2 + \frac{1-2\sigma}{(2\sigma-d)(d-1)} (\partial\psi)^2 \right. \\
\left. - \frac{1}{4} e^{\frac{2(\zeta+\psi)}{(2\sigma-d)} + \frac{2\psi}{d-1}} F_{MN} F^{MN} + e^{-\frac{2\psi}{d-1}} 2\sigma(2\sigma-1) \right].
\end{aligned} \tag{3.1.19}$$

Note that the potential is clearly independent of the scalar ζ . In order to obtain canonically normalized scalar kinetic terms we rescale the scalars as

$$\psi = \sqrt{\frac{(2\sigma-d)(d-1)}{2(2\sigma-1)}} \bar{\psi}, \quad \zeta = \sqrt{\frac{(2\sigma-d)(2\sigma-d-1)}{2}} \bar{\zeta}, \tag{3.1.20}$$

to yield the action in the Einstein frame

$$\begin{aligned}
S_{(d+1)} = L \int d^{d+1}x \sqrt{-\bar{g}_{(d+1)}} \left[\bar{R} - \frac{1}{2}(\partial\bar{\psi})^2 - \frac{1}{2}(\partial\bar{\zeta})^2 + 2\sigma(2\sigma-1)e^{-\bar{\psi}} \sqrt{\frac{2(2\sigma-d)}{(d-1)(2\sigma-1)}} \right. \\
\left. - \frac{1}{4} e^{\sqrt{\frac{2(2\sigma-1)}{(d-1)(2\sigma-d)}} \bar{\psi} + \sqrt{\frac{2(2\sigma-d-1)}{2\sigma-d}} \bar{\zeta}} F_{MN} F^{MN} \right].
\end{aligned} \tag{3.1.21}$$

Note that this rescaling implicitly assumes that $2\sigma > (d+1)$: the scalar ζ has a negative kinetic term whenever $2\sigma < (d+1)$ and therefore cannot be canonically normalized. For

such values of the parameter σ , one would not expect that the scalar ζ is part of a physical compactification, as we discuss next⁸.

3.1.1 Brane interpretation

In this section we discuss whether the $(d+1)$ -dimensional action (3.1.21) can be interpreted in terms of consistent truncations of sphere reductions of decoupled Dp -brane, M-brane and string solutions. Let us begin by reviewing the case with the metric and one scalar, ψ , discussed in [24, 22]. If one truncates the action (3.1.21) to just these fields, and sets

$$2\sigma = d + \frac{(p-3)^2}{(5-p)}, \quad (3.1.22)$$

with $d = p + 1$ and $p \neq 5$, then the action arises from the reduction of the corresponding decoupled p -brane background over a sphere. The scalar field ψ is then dual to the running coupling of the worldvolume theory, whilst the metric is dual to the field theory energy momentum tensor. The general parametrization encompasses the conformal cases of the D3-branes, M2-branes and M5-branes, with the latter M-branes obtained by setting $2\sigma = d$. It also includes the cases of D0-branes, D1-branes, D2-branes, D4-branes and fundamental strings, with the latter corresponding to $p = 1$ in the formula above but excludes five-branes and Dp -branes with $p \geq 6$.

For the non-conformal cases of the D1-branes, D4-branes and fundamental strings,

$$(2\sigma - d) = 1, \quad (3.1.23)$$

which implies that the action (3.1.13) can always be interpreted as an S^1 reduction of a conformal theory. In this case the scalar ζ is not present, as the only reduction is the standard KK reduction over a circle. The gauge field in these cases is just the Kaluza-Klein gauge field of the reduction, corresponding to the conserved current in the reduced field theory.

For the case of D2-branes, notice that

$$(2\sigma - d) = 1/3 < 1, \quad (3.1.24)$$

which implies that the kinetic term (in Einstein frame) in (3.1.13) for the new scalar ζ is negative (or if we work with (3.1.21) the coefficient of F^2 becomes complex and the action is not real). Decoupled D2-branes reduced on an S^6 are believed to admit a consistent truncation [24] which is the $ISO(7)$ gauged supergravity theory [56, 57]. The corresponding operators to these gauged supergravity fields would be the operators in the same supermultiplet as the stress energy tensor. The gauged supergravity theory contains however no scalars with negative kinetic terms, and therefore ζ cannot be interpreted as one of the scalars of the gauged supergravity theory, nor indeed would it seem to have a sensible interpretation in terms of the dual (supersymmetric) gauge theory.

⁸It is interesting to note that a similar action was recently discussed in [55], in the context of p -branes with curved worldvolumes. However, the scalar potentials in this case are different, and one cannot interpret the action given here in terms of branes with curved worldvolumes.

The final standard case is that of the D0-branes, for which

$$(2\sigma - d) = \frac{9}{5}. \quad (3.1.25)$$

In this case the scalar ζ has a positive kinetic term, and there is no a priori obstruction to it being interpreted as one of the scalars arising in an \mathbf{S}^8 compactification of the type IIA theory in ten dimensions. At the same time, there is also no guarantee that the scalar ζ and the gauge field can be identified with fields in the \mathbf{S}^8 reduction.

There is a second natural way to interpret the $(d+1)$ -dimensional actions in terms of decoupled branes and strings: given the action corresponding to a non-conformal p -brane in $d' + 1 \equiv (p+2)$ dimensions, one can always reduce this action on a circle to obtain an action with an additional scalar and gauge field in one less dimension. In such cases, the relation between the index σ and p would be

$$2\sigma = p + 2 + \frac{(p-3)^2}{(5-p)}, \quad (3.1.26)$$

with $p \neq 5$, and the dual theory is the KK reduction of the non-conformal p -brane theory.

3.2 Holographic dictionary

We now want to use the generalized dimensional reduction in order to set up a holographic dictionary for this theory. In general, in order to set up such a dictionary one needs to understand the asymptotic structure of the field equations, which is a hard problem. We have just established however that all solutions of the theory (3.1.21) descend from solutions of (3.1.1) and the most general asymptotic solution of the latter is known:

$$ds_{(2\sigma+1)}^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu} dx^\mu dx^\nu; \quad (3.2.1)$$

$$g_{\mu\nu} = g_{(0)\mu\nu} + \rho g_{(2)\mu\nu} + \cdots + \rho^\sigma (g_{(2\sigma)\mu\nu} + h_{(2\sigma)\mu\nu} \log \rho) + \cdots, \quad (3.2.2)$$

where $g_{(0)\mu\nu}$ is the source, only the trace and divergence of $g_{(2\sigma)\mu\nu}$ are determined locally in terms of the source and all other coefficients are completely determined. The logarithmic terms $h_{(2\sigma)}$ are present only when σ is integral. It follows that it suffices to consider the class of asymptotic solutions that is also of the form (3.1.12) required for the reduction in order to obtain the general asymptotic solution of (3.1.13).

The $(d+1)$ -dimensional metric is expanded in the usual Fefferman-Graham form, as above, whilst the scalar fields can be expanded as

$$\begin{aligned} e^{\frac{2\psi}{(2\sigma-d)}} &= \frac{1}{\rho} e^{\frac{2\kappa}{(2\sigma-d)}}; & \kappa &= \kappa_{(0)} + \rho \kappa_{(2)} + \cdots + \rho^\sigma \kappa_{(2\sigma)}; \\ \zeta &= \zeta_{(0)} + \rho \zeta_{(2)} + \cdots + \rho^\sigma \zeta_{(2\sigma)}, \end{aligned} \quad (3.2.3)$$

and the gauge field as

$$A_i(\rho, z) = A_{i(0)}(z) + \rho A_{i(2)}(z) + \cdots + \rho^\sigma A_{i(2\sigma)}(z) + \cdots \quad (3.2.4)$$

There are log terms present when σ is an integer, but these are once again suppressed, as we are primarily interested in the cases where σ is non-integral. Here $\kappa_{(0)}$, $\zeta_{(0)}$ and $A_{(0)}$ are the sources to dual scalar operators, \mathcal{O}_ψ , \mathcal{O}_ζ and the conserved current J^i . The subleading coefficients are locally related to the sources, up to the order where the vev of the dual operator appears. The precise form can be worked out from the known local relation between the subleading coefficients in (3.2.1) and $g_{(0)}$ (see appendix A of [18]), but we will not need these relations here.

Having obtained the asymptotic solution, one would then next like to compute the local boundary counterterms that would render finite the on-shell action. Happily, this can be easily done using the generalized dimensional reduction [25]. Given σ we choose any half-integer $\tilde{\sigma} > \sigma$ and determine the $[\sigma] + 1$ most singular $\text{AdS}_{(2\tilde{\sigma}+1)}$ -counterterms as a function of $\tilde{\sigma}$, where $[\sigma]$ denotes the largest integer less than or equal to σ (when σ is an integer one of these counterterms is logarithmic). Reducing these $\text{AdS}_{(2\tilde{\sigma}+1)}$ -counterterms and replacing $\tilde{\sigma}$ by σ yields the counterterms appropriate for (3.1.13).

As an example let us consider the counterterm action for $1 < \sigma < 2$, for which we only need two counterterms. The two most singular counterterms in $\text{AdS}_{2\tilde{\sigma}+1}$ defined on a regulating hypersurface are given by (see appendix B of [18])⁹

$$S_{(2\tilde{\sigma})}^{ct} = L_{AdS} \int_{\rho=\epsilon} d^{2\tilde{\sigma}} x \sqrt{-\gamma_{2\tilde{\sigma}}} \left[2(2\tilde{\sigma} - 1) + \frac{1}{2\tilde{\sigma} - 2} \hat{R}[\gamma_{2\tilde{\sigma}}] \right], \quad (3.2.5)$$

where $\gamma_{2\tilde{\sigma}ij}$ is the induced metric on the $(2\tilde{\sigma})$ -dimensional hypersurface and $\hat{R}[\gamma_{2\tilde{\sigma}}]$ the corresponding curvature. The counterterm action to (3.1.13) for $1 < \sigma < 2$ is then given by reducing (3.2.5) to d dimensions and replacing $\tilde{\sigma}$ with σ ,

$$S_{(d)}^{ct} = L \int_{\rho=\epsilon} d^d x \sqrt{-\gamma_d} e^\psi \left[2(2\sigma - 1) + \frac{1}{2\sigma - 2} \left(\hat{R}_d + \frac{2\sigma - d - 1}{2\sigma - d} (\partial\psi)^2 - \frac{1}{(2\sigma - d - 1)(2\sigma - d)} (\partial\zeta)^2 - \frac{1}{4} e^{\frac{2(\zeta+\psi)}{2\sigma-d}} F_{ij} F^{ij} \right) \right]. \quad (3.2.6)$$

This covers early results for $d = 3$, [58]. When $\sigma > 2$ one needs to include additional gravitational counterterms.

Next let us turn to holographic one point functions. These can be computed by functionally differentiating the renormalized on-shell action, S_{ren} , but again the dimensional reduction offers a shortcut: we simply need to reduce the formula for the 1-point function. The latter reads [18],

$$\langle T_{\mu\nu} \rangle_{2\sigma} = \frac{2}{\sqrt{-g_{(0),2\sigma}}} \frac{\delta S_{ren}}{\delta g_{(0)}^{\mu\nu}} = 2\sigma L_{AdS} g_{(2\sigma)\mu\nu} + \dots, \quad (3.2.7)$$

where the ellipses denote terms that locally depend on $g_{(0)\mu\nu}$. These terms are present when $g_{(0)\mu\nu}$ is curved and there is a conformal anomaly, *i.e.* when σ is an integer. They do not play an important role in the discussion here and so they will be suppressed.

⁹Note that convention for the curvature tensor used in [18] has the opposite sign.

Reduction of the expectation value of the higher-dimensional stress energy tensor gives the expectation values of the operators in the d -dimensional field theory. Let us begin by writing the former in terms of components longitudinal and transverse to the reduction torus. When we do so, we should also take into account the additional prefactor $(2\pi R)^{2\sigma-d}$ of the lower-dimensional action in (3.1.3) which results from the integration over the torus and for the change in the determinant of the metric in the definition of the vev, $\sqrt{g_{(0),d}} = e^{-\kappa_{(0)}} \sqrt{g_{(0),2\sigma}}$. To accommodate these factors, we define

$$\langle t_{\mu\nu} \rangle_d \equiv e^{\kappa_{(0)}} (2\pi R)^{2\sigma-d} \langle T_{\mu\nu} \rangle_{2\sigma}. \quad (3.2.8)$$

Then one obtains

$$\begin{aligned} \langle t_{ij} \rangle_d &= 2\sigma L \left[e^{\kappa_{(0)}} g_{(2\sigma)ij} + 2e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}} \left(A_{i(0)} A_{j(2\sigma)} + \frac{A_{i(0)} A_{j(0)}}{2\sigma-d} (\kappa_{(2\sigma)} + \zeta_{(2\sigma)}) \right) \right], \\ \langle t_{iy} \rangle_d &= -2\sigma L e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}} \left(A_{i(2\sigma)} + \frac{2}{2\sigma-d} (\kappa_{(2\sigma)} + \zeta_{(2\sigma)}) A_{i(0)} \right), \\ \langle t_{yy} \rangle_d &= \frac{4\sigma L}{(2\sigma-d)} e^{\frac{(2\sigma-d+2)\kappa_{(0)}+2\zeta_{(0)}}{2\sigma-d}} (\kappa_{(2\sigma)} + \zeta_{(2\sigma)}) + \dots \equiv -e^{\frac{2}{(2\sigma-d)}(\kappa_{(0)}+\zeta_{(0)})} \langle \mathcal{O}_1 \rangle_d, \\ \langle t_{ab} \rangle_d &= \frac{4\sigma L}{2\sigma-d} e^{\frac{1}{2\sigma-d}((2\sigma-d+2)\kappa_{(0)}-\frac{2}{(2\sigma-d-1)}\zeta_{(0)})} \left(\kappa_{(2\sigma)} - \frac{1}{(2\sigma-d-1)} \zeta_{(2\sigma)} \right) \delta_{ab} + \dots \\ &\equiv -e^{\frac{2}{2\sigma-d}(\kappa_{(0)}-\frac{1}{(2\sigma-d-1)}\zeta_{(0)})} \langle \mathcal{O}_2 \rangle_d \delta_{ab}, \end{aligned} \quad (3.2.9)$$

where the ellipses again contain curvatures of the boundary metric $g_{(0)ij}$ and derivatives of $(\kappa_{(0)}, \zeta_{(0)})$. From these expressions we read off

$$\begin{aligned} \langle \mathcal{O}_1 \rangle_d &= -\frac{4\sigma L}{2\sigma-d} e^{\kappa_{(0)}} (\kappa_{(2\sigma)} + \zeta_{(2\sigma)}) + \dots, \\ \langle \mathcal{O}_2 \rangle_d &= -\frac{4\sigma L}{2\sigma-d} e^{\kappa_{(0)}} \left(\kappa_{(2\sigma)} - \frac{1}{(2\sigma-d-1)} \zeta_{(2\sigma)} \right) + \dots \end{aligned} \quad (3.2.10)$$

The reduction gives, as expected, a symmetric tensor operator t_{ij} , a vector operator t_{iy} and two scalar operators. The normalizations of all the operators at this point is somewhat arbitrary with the stress energy tensor, current and naturally normalized scalar operators of the dual d -dimensional field theory being formed from linear combinations of these operators. The combinations which form the d -dimensional field theory operators can be obtained by varying the renormalized onshell actions with respect to the appropriate sources, for example the stress energy tensor follows from varying the action with respect to the d -dimensional metric source. There is however a simple way to deduce the appropriate combinations from the reduction of the higher-dimensional Ward identities. Anticipating how this reduction will work, let us introduce linear combinations of the scalar operators such that

$$\begin{aligned} \langle \mathcal{O}_\psi \rangle_d &= \frac{1}{(2\sigma-d)} [(2\sigma-d-1)\langle \mathcal{O}_2 \rangle_d + \langle \mathcal{O}_1 \rangle_d]; \\ \langle \mathcal{O}_\zeta \rangle_d &= \frac{1}{(2\sigma-d)} [\langle \mathcal{O}_1 \rangle_d - \langle \mathcal{O}_2 \rangle_d]. \end{aligned} \quad (3.2.11)$$

The notation follows from the fact that the field ψ will act as a source for \mathcal{O}_ψ whilst the field ζ sources \mathcal{O}_ζ . It is useful to recall the $\zeta = 0$ limit. In this case note that $\langle \mathcal{O}_1 \rangle_d = \langle \mathcal{O}_2 \rangle_d$, with the operator $\langle \mathcal{O}_\phi \rangle_d$ defined in [22] taking the expectation value

$$\langle \mathcal{O}_\phi \rangle_d = \langle \mathcal{O}_1 \rangle_d = \langle \mathcal{O}_2 \rangle_d. \quad (3.2.12)$$

The conformal Ward identity $\langle T_\mu^\mu \rangle_{2\sigma} = \mathcal{A}_{2\sigma}$ in the 2σ -dimensional theory can be reduced to

$$\begin{aligned} & \langle t_i^i \rangle_d - 2A_{(0)}^i \langle t_{iy} \rangle_d - (2\sigma - d - 1) \langle \mathcal{O}_2 \rangle_d - \left(1 + e^{\frac{2(\kappa_{(0)} + \zeta_{(0)})}{2\sigma - d}} A_{(0)i} A_{(0)}^i \right) \langle \mathcal{O}_1 \rangle_d \\ &= e^{\kappa_{(0)}} (2\pi R)^{2\sigma - d} \mathcal{A}_{2\sigma} \equiv \mathcal{A}_d. \end{aligned} \quad (3.2.13)$$

Furthermore, if we write

$$\langle J_i \rangle_d = \langle t_{iy} \rangle_d + A_{(0)i} \langle t_{yy} \rangle_d, \quad (3.2.14)$$

$$\langle T_{ij} \rangle_d = \langle t_{ij} \rangle_d + (A_{(0)i} \langle J_j \rangle + A_{(0)j} \langle J_i \rangle) + A_{(0)i} A_{(0)j} e^{\frac{2(\kappa_{(0)} + \zeta_{(0)})}{2\sigma - d}} \langle \mathcal{O}_1 \rangle_d,$$

so that

$$\begin{aligned} \langle J_i \rangle_d &= -2\sigma L e^{\frac{1}{2\sigma - d}((2\sigma - d + 2)\kappa_{(0)} + 2\zeta_{(0)})} A_{(2\sigma)i} + \dots; \\ \langle T_{ij} \rangle_d &= 2\sigma L e^{\kappa_{(0)}} g_{(2\sigma)ij} + \dots, \end{aligned} \quad (3.2.15)$$

the dilatation Ward identity becomes simply

$$\langle T_i^i \rangle_d - (2\sigma - d) \langle \mathcal{O}_\psi \rangle_d = \mathcal{A}_d. \quad (3.2.16)$$

Note in particular that the new scalar operator \mathcal{O}_ζ does not contribute to the dilatation Ward identity.

Using these linear combinations of the operators, the conservation equation for the higher-dimensional stress energy tensor reduces to

$$\nabla^i \langle T_{ij} \rangle_d + \partial_j \kappa_{(0)} \langle \mathcal{O}_\psi \rangle_d + \partial_j \zeta_{(0)} \langle \mathcal{O}_\zeta \rangle_d - F_{(0)j}^i \langle J_i \rangle_d = 0, \quad (3.2.17)$$

and the divergence equation for a current

$$\nabla^i \langle J_i \rangle_d = 0. \quad (3.2.18)$$

Looking at the first divergence equation we can recognize it as the standard diffeomorphism Ward identity for a theory with stress energy tensor T_{ij} in which the other operators are defined in terms of the generating functional W

$$\langle J^i \rangle_d = -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta A_{(0)i}}; \quad \langle \mathcal{O}_\psi \rangle_d = -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta \kappa_{(0)}}; \quad \langle \mathcal{O}_\zeta \rangle_d = -\frac{1}{\sqrt{g_{(0)}}} \frac{\delta W}{\delta \zeta_{(0)}}, \quad (3.2.19)$$

indicating that the non-normalizable modes of (ψ, ζ) do indeed source $(\mathcal{O}_\psi, \mathcal{O}_\zeta)$ respectively, as anticipated, whilst $A_{(0)i}$ sources the conserved current J^i . One could directly verify these relations by varying the renormalized bulk onshell action.

3.3 Black branes

In preparation of our discussion of hydrodynamics in the next subsection, we will now discuss a realization of the setup in the previous section using black branes. Recall that conformal hydrodynamics was derived in [59] by studying the long wavelength fluctuation equations around the boosted black D3 brane geometry. The universal hydrodynamics related to the non-conformal branes is similarly related to long wavelength fluctuation equations around the boosted black Dp brane geometry and can be most easily obtained by starting from the (conformal) black brane solution in $(2\sigma+1)$ dimensions and then carrying out the generalized dimensional reduction [25]. Here the starting point is a (conformal) black brane solution in $(2\sigma+1)$ dimensions:

$$\begin{aligned} ds_{(2\sigma+1)}^2 &= \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho} [-f(\rho) dt'^2 + dy'^2 + dz_r dz^r + dy_a dy^a], \\ f(\rho) &= 1 - m^{2\sigma} \rho^\sigma, \end{aligned} \quad (3.3.1)$$

where (y, y^a, z^r) run over all transverse coordinates ($a = d+1, \dots, 2\sigma-1$). This metric is Einstein with negative curvature when 2σ is an integer, and has an event horizon at $\rho = m^{-2}$. The Hawking temperature T and Bekenstein-Hawking entropy density s are given by

$$T = \frac{m\sigma}{2\pi}, \quad s = 4\pi L_{AdS} m^{2\sigma-1}. \quad (3.3.2)$$

Performing a Lorentz transformation $t = \cosh \omega t' - \sinh \omega y'$, $y = \cosh \omega y' - \sinh \omega t'$, the resulting metric can carry a wave:

$$\begin{aligned} ds_{(2\sigma+1)}^2 &= \frac{d\rho^2}{4\rho^2 f(\rho)} - \rho^{-1} K(\rho)^{-1} f(\rho) dt^2 + \frac{K(\rho)}{\rho} [dy - ((K'(\rho))^{-1} - 1) dt]^2 \\ &\quad + \rho^{-1} dz_r dz^r + \rho^{-1} dy_a dy^a, \\ f(\rho) &= 1 - m^{2\sigma} \rho^\sigma, \quad K(\rho) = (1 + Q\rho^\sigma), \\ (K'(\rho))^{-1} &= (1 - \bar{Q}\rho^\sigma K(\rho)^{-1}), \end{aligned} \quad (3.3.3)$$

where

$$Q = m^{2\sigma} \sinh^2 \omega; \quad \bar{Q} = m^{2\sigma} \sinh \omega \cosh \omega. \quad (3.3.4)$$

Setting $\omega = 0$ removes the wave, whilst the extremal limit is recovered in the limit $m \rightarrow 0$ with $\omega \rightarrow \infty$ and Q finite. When $(2\sigma+1)$ is integral, this solution arises from a standard non-extremal intersection of one of the conformal branes (D3, M2, M5) with a wave (see for example [60, 61]), taking a decoupling limit (which focuses the geometry near the brane) and then reducing over the transverse sphere. The physical interpretation of cases in which $(2\sigma+1)$ is non-integral will be discussed in the next section.

With a view towards dimensional reduction, we consider now the coordinates (y, y^a) periodically identified with period $2\pi R$ (as in subsection (3.1)), and rewrite the geometry as

$$\begin{aligned} ds_{(2\sigma+1)}^2 &= \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho} (dz^r dz_r - dt^2) + \frac{1}{\rho} (1 - K(\rho)^{-1} f(\rho)) dt^2 \\ &\quad + \frac{1}{\rho} dy^a dy_a + \frac{K(\rho)}{\rho} [dy - ((K'(\rho))^{-1} - 1) dt]^2. \end{aligned} \quad (3.3.5)$$

Now let us boost this geometry along the non-compact boundary dimensions with boost parameter \hat{u}_i (where now $z^i = (t, z^r)$, *i.e.* the y and y^a directions are excluded). This results in

$$\begin{aligned} ds_{(2\sigma+1)}^2 = & \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho} (dz^i dz_i) + \frac{1}{\rho} (1 - K(\rho)^{-1} f(\rho)) \hat{u}_i \hat{u}_j dz^i dz^j \\ & + \frac{1}{\rho} dy^a dy_a + \frac{K(\rho)}{\rho} [dy - ((K'(\rho))^{-1} - 1) \hat{u}_i dz^i]^2. \end{aligned} \quad (3.3.6)$$

Note that the fluid velocity \hat{u}^i does *not* square to -1 with the 2σ -dimensional boundary metric, but $\eta_{ij} \hat{u}^i \hat{u}^j = -1$. In what follows, we shall also include an external, uniform gauge field $A_{(0)i} dz^i$, which can be obtained from (3.3.6) by performing a coordinate transformation on y as $dy \rightarrow dy + A_{(0)i} dz^i$. Once we allow the temperature, charge, fluid velocity and external gauge field to become position dependent the metric needs to be corrected at each order to satisfy the field equations.

We now reduce (3.3.6). The reduced metric is then

$$ds_{(d+1)}^2 = \frac{d\rho^2}{4\rho^2 f(\rho)} + \frac{1}{\rho} (dz^i dz_i) + \frac{1}{\rho} (1 - K(\rho)^{-1} f(\rho)) \hat{u}_i \hat{u}_j dz^i dz^j, \quad (3.3.7)$$

with the scalar fields being

$$e^{\frac{2\phi_2}{(2\sigma-d-1)}} = \frac{1}{\rho}, \quad e^{2\phi_1} = \frac{K(\rho)}{\rho}, \quad (3.3.8)$$

and the gauge field is

$$A = [A_{(0)i} + ((K'(\rho))^{-1} - 1) \hat{u}_i] dz^i. \quad (3.3.9)$$

Rewriting the scalar fields in terms of (ψ, ζ) we obtain

$$e^\psi = \frac{1}{\rho^{\sigma-d/2}} K(\rho)^{1/2}, \quad e^\zeta = K(\rho)^{\frac{1}{2}(2\sigma-d-1)}. \quad (3.3.10)$$

It is useful to rewrite quantities using Fefferman-Graham coordinates (see [25] for the derivation of the coordinate transformation). The reduced metric is then

$$\begin{aligned} ds_{(d+1)}^2 = & \frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \left(1 + \frac{m^{2\sigma} \tilde{\rho}^\sigma}{4} \right)^{\frac{2}{\sigma}} dz_i dz^i \\ & + \frac{1}{\tilde{\rho}} \left(1 + \frac{m^{2\sigma} \tilde{\rho}^\sigma}{4} \right)^{\frac{2}{\sigma}} [1 - K(\rho(\tilde{\rho}))^{-1} f(\rho(\tilde{\rho}))] \hat{u}_i \hat{u}_j dz^i dz^j. \end{aligned} \quad (3.3.11)$$

with the scalar fields being

$$e^{\frac{2\phi_2}{(2\sigma-d-1)}} = \frac{1}{\tilde{\rho}} \left(1 + \frac{m^{2\sigma} \tilde{\rho}^\sigma}{4} \right)^{\frac{2}{\sigma}}, \quad e^{2\phi_1} = \frac{K(\rho(\tilde{\rho}))}{\tilde{\rho}} \left(1 + \frac{m^{2\sigma} \tilde{\rho}^\sigma}{4} \right)^{\frac{2}{\sigma}}, \quad (3.3.12)$$

and the gauge field is

$$A = A_{(0)i} dz^i + \left[(K'(\rho(\tilde{\rho})))^{-1} - 1 \right] \hat{u}_i dz^i. \quad (3.3.13)$$

Again rewriting the scalar fields in terms of (ψ, ζ) we obtain

$$e^\psi = \frac{K(\rho(\tilde{\rho}))^{\frac{1}{2}}}{\tilde{\rho}^{\frac{2\sigma-d}{2}}} \left(1 + \frac{m^{2\sigma}\tilde{\rho}^\sigma}{4}\right)^{\frac{2\sigma-d}{\sigma}}, \quad e^\zeta = K(\rho(\tilde{\rho}))^{\frac{2\sigma-d-1}{2}}. \quad (3.3.14)$$

Now since

$$e^{\frac{2\psi}{2\sigma-d}} = \frac{1}{\tilde{\rho}} e^{\frac{2\kappa}{2\sigma-d}}$$

we get that

$$e^\kappa = K(\rho(\tilde{\rho}))^{\frac{1}{2}} \left(1 + \frac{m^{2\sigma}\tilde{\rho}^\sigma}{4}\right)^{\frac{2\sigma-d}{\sigma}}. \quad (3.3.15)$$

We can expand the above results in $\tilde{\rho}$ to get

$$\begin{aligned} \kappa_{(0)} &= 0 ; \quad \kappa_{(2\sigma)} = \frac{1}{2}Q + \frac{2\sigma-d}{\sigma} \frac{m^{2\sigma}}{4}, \\ \zeta_{(0)} &= 0 ; \quad \zeta_{(2\sigma)} = \frac{2\sigma-d-1}{2}Q, \\ A_{i(2\sigma)} &= \hat{u}_i \bar{Q}. \end{aligned} \quad (3.3.16)$$

The source for the gauge field is $A_{i(0)}$, i.e. the term in square brackets in (3.3.13) goes to zero as $\tilde{\rho} \rightarrow 0$.

These allow us to extract the expectation values of the dual operators using (3.2.9). One finds,

$$\begin{aligned} \langle T_{ij} \rangle_d &= Lm^{2\sigma} \eta_{ij} + 2\sigma L(Q + m^{2\sigma}) \hat{u}_i \hat{u}_j \\ &= Lm^{2\sigma} (\eta_{ij} + 2\sigma \cosh^2 \omega \hat{u}_i \hat{u}_j); \\ \langle J_i \rangle_d &= 2\sigma L \bar{Q} \hat{u}_i \\ &= 2\sigma Lm^{2\sigma} \sinh \omega \cosh \omega \hat{u}_i; \\ \langle \mathcal{O}_1 \rangle_d &= -m^{2\sigma} L - 2\sigma LQ \\ &= -Lm^{2\sigma} (1 + 2\sigma \sinh^2 \omega) \\ \langle \mathcal{O}_2 \rangle_d &= -Lm^{2\sigma}; \end{aligned} \quad (3.3.17)$$

which one can verify indeed satisfies the dilatation Ward identity (3.2.16). From these expressions we can also read off the thermodynamic quantities,

$$\hat{\epsilon} = Lm^{2\sigma}(2\sigma \cosh^2 \omega - 1), \quad \hat{q} \equiv \bar{Q} = 2\sigma Lm^{2\sigma} \sinh \omega \cosh \omega, \quad \hat{P} = Lm^{2\sigma}, \quad (3.3.18)$$

where $\hat{\epsilon}$ is the energy density, \hat{q} the charge density and \hat{P} the pressure of the reduced spacetime (3.3.11). One may solve the first two equations to express m and ω in terms of $\hat{\epsilon}$ and \hat{q} and then use them in the last relation to obtain the equation of state $\hat{P} = \hat{P}(\hat{\epsilon}, \hat{q})$,

$$\hat{P}(\hat{\epsilon}, \hat{q}) = \frac{1}{2\sigma-1} \left(\sqrt{\hat{\epsilon}^2(\sigma-1)^2 + (\hat{\epsilon}^2 - \hat{q}^2)(2\sigma-1)} - \hat{\epsilon}(\sigma-1) \right). \quad (3.3.19)$$

(Since $\hat{P} = Lm^{2\sigma}$ this relation also expresses m in terms of $\hat{\epsilon}, \hat{q}$, while $\sinh 2\omega = \hat{q}/(\sigma\hat{P})$ gives ω in terms of $\hat{\epsilon}$ and \hat{q}). In the limit $\hat{q} \rightarrow 0$ we get the equation of state for the

non-conformal branes. Above extremality, $\hat{\epsilon} > |\hat{q}|$, the expression under the square root is manifestly positive. In the extremal limit $\hat{\epsilon} \rightarrow |\hat{q}|$ the pressure vanishes, as expected. The reduced temperature and entropy are

$$\hat{T} = \frac{m\sigma}{2\pi \cosh \omega}, \quad \hat{s} = 4\pi L \cosh \omega m^{2\sigma-1}. \quad (3.3.20)$$

From the equation of state (3.3.19), we may obtain the adiabatic speed of sound¹⁰

$$\hat{c}_s^2 = \left. \frac{\partial \hat{P}}{\partial \hat{\epsilon}} \right|_{\hat{s}/\hat{q}}, \quad (3.3.21)$$

keeping fixed the ratio \hat{s}/\hat{q} . Using (3.3.20) and (3.3.18), this yields

$$d\left(\frac{\hat{s}}{\hat{q}}\right) = 0 \Rightarrow d\omega = -\tanh \omega \frac{dm}{m} \quad (3.3.22)$$

so that

$$\hat{c}_s^2 = \frac{1}{2(\sigma-1) \cosh^2 \omega + 1} \quad (3.3.23)$$

which reduces to the result for the neutral black branes derived in [25].

Furthermore, from (3.3.9) we obtain that the chemical potential is equal to

$$\hat{\mu} = -\left(\hat{u}^i A_i|_{\rho=0} - \hat{u}^i A_i|_{\rho=m^{-2}}\right) = \tanh \omega. \quad (3.3.24)$$

Regularity at the horizon requires that $\hat{u}^i A_i|_{\rho=m^{-2}} = 0$ which then fixes the external gauge field in terms of the chemical potential. We will however relax this condition so that we can incorporate a general external gauge field in the next subsection. We note, however, that all of the main results (transport coefficients etc.) can equally be obtained without turning on an additional external field beyond that required by the presence of the chemical potential. One may also verify that the thermodynamic identities,

$$\hat{P} + \hat{\epsilon} = \hat{T}\hat{s} + \hat{q}\hat{\mu}, \quad d\hat{P} = \hat{s}d\hat{T} + \hat{q}d\hat{\mu} \quad (3.3.25)$$

hold.

It is interesting to observe that the expectation values of the scalar operators, $(\langle \mathcal{O}_\psi \rangle_d, \langle \mathcal{O}_\zeta \rangle_d)$, can be expressed in terms of the energy density and pressure as

$$\begin{aligned} \langle \mathcal{O}_\psi \rangle_d &= \frac{1}{(2\sigma-d)} \langle T_i^i \rangle_d = \frac{1}{(2\sigma-d)} \left[(d-1)\hat{P} - \hat{\epsilon} \right]; \\ \langle \mathcal{O}_\zeta \rangle_d &= \frac{1}{(2\sigma-d)} \left[(2\sigma-1)\hat{P} - \hat{\epsilon} \right]. \end{aligned} \quad (3.3.26)$$

Thus the expectation value of the scalar operator $\langle \mathcal{O}_\psi \rangle_d$ characterizes the deviation of the equation of state from conformality (as one would expect) whilst the expectation value of the second operator $\langle \mathcal{O}_\zeta \rangle_d$ is zero in the uncharged case, in which case the equation of state indeed reduces to that of the non-conformal branes, $\hat{P} = \hat{\epsilon}/(2\sigma-1)$.

¹⁰See [62], Chapter XV, equation (134.14) and (134.7).

3.4 Universal Hydrodynamics

We would now like to use the generalized dimensional reduction in order to obtain the universal hydrodynamics corresponding to the charged dilatonic solutions. Recall that the hydrodynamic energy-momentum tensor for a conformal fluid at first-derivative order in (2σ) dimensions on a curved manifold with metric $g_{(0)\mu\nu}$ is

$$\begin{aligned} \langle T_{\mu\nu} \rangle_{2\sigma} &= \langle T_{\mu\nu}^{\text{eq}} \rangle_{2\sigma} + \langle T_{\mu\nu}^{\text{diss}} \rangle_{2\sigma} \\ \langle T_{\mu\nu}^{\text{eq}} \rangle_{2\sigma} &= P(g_{(0)\mu\nu} + 2\sigma u_\mu u_\nu), \quad \langle T_{\mu\nu}^{\text{diss}} \rangle_{2\sigma} = -2\eta_{2\sigma} \sigma_{\mu\nu}, \\ \sigma_{\mu\nu} &= P_\mu^\kappa P_\nu^\lambda \nabla_{(\kappa} u_{\lambda)} - \frac{1}{2\sigma - 1} P_{\mu\nu} (\nabla \cdot u), \quad P_{\mu\nu} = g_{(0)\mu\nu} + u_\mu u_\nu, \end{aligned} \quad (3.4.1)$$

where T , u_μ and $\eta_{2\sigma}$ denote the temperature, velocity and shear viscosity respectively of the fluid and ∇_μ is the covariant derivative corresponding to the metric $g_{(0)\mu\nu}$. Note that we are working in Landau-Lifshitz frame,

$$u^\mu \langle T_{\mu\nu}^{\text{diss}} \rangle_{2\sigma} = 0. \quad (3.4.2)$$

The evolution of the fluid is determined by the conservation of the energy-momentum tensor,

$$\nabla^\mu \langle T_{\mu\nu} \rangle_{2\sigma} = 0. \quad (3.4.3)$$

For the AdS black brane,

$$P = L_{\text{AdS}} m^{2\sigma}, \quad \eta_{2\sigma} = \frac{s}{4\pi} = L_{\text{AdS}} m^{2\sigma-1} \quad (3.4.4)$$

by (3.3.2).

Let us first determine the reduced fluid velocity. The boundary metric can be read off the reduction Ansatz (3.1.12), using the expansions of the fields (3.2.1), (3.2.3) and (3.2.4). For simplicity, we set $\kappa_{(0)} = \zeta_{(0)} = 0$ as in the case of the AdS black brane (3.3.16). Then,

$$g_{(0)ij} = \eta_{ij} + A_{(0)i} A_{(0)j}, \quad g_{(0)iy} = -A_{(0)i}, \quad g_{(0)yy} = 1, \quad (3.4.5)$$

and the inverse metric is given by

$$g_{(0)}^{ij} = \eta^{ij}, \quad g_{(0)}^{iy} = A_{(0)}^i, \quad g_{(0)}^{yy} = 1 + \eta^{ij} A_{(0)i} A_{(0)j}. \quad (3.4.6)$$

Note that the reduced boundary metric is simply the Minkowski metric η_{ij} ¹¹. One may then derive the reduced fluid velocity \hat{u}^i by requiring that both

$$u^\mu u_\mu = -1, \quad u^\mu = g_{(0)}^{\mu\nu} u_\nu, \quad (3.4.7)$$

and

$$\hat{u}^i \hat{u}_i = -1, \quad \hat{u}^i = \eta^{ij} \hat{u}_j, \quad (3.4.8)$$

¹¹The hydrodynamics at first-order is independent of the curvature of the reduced boundary metric, so our results will still hold at first-order for a curved boundary in d dimensions.

It is convenient to choose by convention (and to make a link with the wave generating coordinate transformation of the previous subsection)

$$u_y = \sinh \omega \quad (3.4.9)$$

so that, setting $u_a = 0$ along the remaining compact dimensions y^a ,

$$u_i = \cosh \omega \hat{u}_i - \sinh \omega A_{(0)i}, \quad u_y = \sinh \omega, \quad u^i = \cosh \omega \hat{u}^i, \quad u^y = \sinh \omega + \cosh \omega \hat{u} \cdot \partial A_{(0)}. \quad (3.4.10)$$

We may now turn to the equilibrium part. Inserting in $\langle T_{\mu\nu}^{\text{eq}} \rangle_{2\sigma}$ and using (3.2.14) we obtain:

$$\begin{aligned} \langle T_{ij}^{\text{eq}} \rangle_d &= \hat{P} [\eta_{ij} + 2\sigma (u_i + u_y A_{(0)i}) (u_j + u_y A_{(0)j})], \\ \langle J_i^{\text{eq}} \rangle_d &= 2\sigma \hat{P} u_y (u_i + u_y A_{(0)i}), \\ \langle \mathcal{O}_1^{\text{eq}} \rangle_d &= -\hat{P} (1 + 2\sigma u_y^2), \\ \langle \mathcal{O}_2^{\text{eq}} \rangle_d \delta_{ab} &= -\hat{P} (\delta_{ab} + 2\sigma u_a u_b). \end{aligned}$$

Using (3.4.10), these become

$$\langle T_{ij}^{\text{eq}} \rangle_d = \hat{P} (\eta_{ij} + 2\sigma \cosh^2 \omega \hat{u}_i \hat{u}_j), \quad (3.4.11)$$

$$\langle J_i^{\text{eq}} \rangle_d = 2\sigma \sinh \omega \cosh \omega \hat{P} \hat{u}_i, \quad (3.4.12)$$

$$\langle \mathcal{O}_1^{\text{eq}} \rangle_d = -\hat{P} (1 + 2\sigma \sinh^2 \omega), \quad (3.4.13)$$

$$\langle \mathcal{O}_2^{\text{eq}} \rangle_d = -\hat{P}, \quad (3.4.14)$$

$$\langle \mathcal{O}_\psi^{\text{eq}} \rangle_d = -\frac{2 \sinh^2 \omega \hat{P}}{2\sigma - d}, \quad (3.4.15)$$

$$\langle \mathcal{O}_\zeta^{\text{eq}} \rangle_d = -\frac{\hat{P}}{2\sigma - d} (2\sigma \cosh^2 \omega - d), \quad (3.4.16)$$

so that the equilibrium quantities are

$$\hat{P} = \frac{L}{L_{AdS}} P, \quad \hat{\epsilon} = (2\sigma \cosh^2 \omega - 1) \hat{P}, \quad \hat{q} = 2\sigma \sinh \omega \cosh \omega \hat{P}. \quad (3.4.17)$$

Inserting the value of the pressure density (3.4.4) for the AdS black brane allows to recover the correct reduced pressure, energy and charge density (3.3.18) as well as the dual operators (3.3.17) and (3.3.26).

Let us now discuss the dissipative part. We simply need to insert $u^\mu = (u^i, 0, u^y)$ in $\langle T_{\mu\nu}^{\text{diss}} \rangle_{2\sigma}$ and reduce to d dimensions. The Landau-Lifshitz frame condition (3.4.2) becomes in the reduced theory

$$\begin{aligned} \hat{u}^i \langle J_i^{\text{diss}} \rangle_d &= \tanh \omega \langle \mathcal{O}_1^{\text{diss}} \rangle_d \\ \hat{u}^i \langle T_{ij}^{\text{diss}} \rangle_d &= -\tanh \omega \langle J_j^{\text{diss}} \rangle_d \end{aligned} \quad (3.4.18)$$

and in particular one finds out that the reduced frame is *not* in the Landau frame, so that some care is needed to extract the transport coefficients: we will use the frame independent

formulation discussed in [63]. The authors derive them without assuming any choice of frame but using invariance and symmetry considerations. This approach is thus well-suited to our case, since upon reduction one does not end naturally in the Landau or Eckhart frame. The method relies on ensuring that the divergence of the entropy current is positive semi-definite. In $2\sigma + 1$ dimensions, the entropy current expressed in the Landau frame is

$$\langle J_s^\mu \rangle_{2\sigma} = s u^\mu, \quad (3.4.19)$$

and it obeys the divergence equation (see for example [63]),

$$\nabla_\mu \langle J_s^\mu \rangle_{2\sigma} = -\nabla_\mu \left(\frac{u_\nu}{T} \right) \langle T_{diss}^{\mu\nu} \rangle_{2\sigma} = -\frac{1}{T} \sigma_{\mu\nu} \langle T_{diss}^{\mu\nu} \rangle_{2\sigma}. \quad (3.4.20)$$

For this to be positive semi-definite the shear viscosity, $\eta_{2\sigma}$,

$$P_\mu^\kappa P_\nu^\lambda \langle T_{\kappa\lambda}^{\text{diss}} \rangle_{2\sigma} - \frac{1}{2\sigma - 1} P_{\mu\nu} P^{\kappa\lambda} \langle T_{\kappa\lambda}^{\text{diss}} \rangle_{2\sigma} = -2\eta_{2\sigma} \sigma_{\mu\nu}. \quad (3.4.21)$$

must be non-negative, $\eta_{2\sigma} \geq 0$.

For charged fluids, the entropy current is given by [63]¹²

$$\langle J_s^i \rangle_d = \hat{s} \hat{u}^i - \frac{\hat{u}_j}{\hat{T}} \langle T_{diss}^{ij} \rangle_d - \frac{\hat{\mu}}{\hat{T}} \langle J_{diss}^i \rangle_d. \quad (3.4.22)$$

Imposing the reduced Landau frame conditions (3.4.18), we find that this coincides with the reduction of the entropy current (3.4.19),

$$\langle J_s^i \rangle_d = \hat{s} \hat{u}^i, \quad \hat{s} = \frac{L \cosh \omega}{L_{AdS}} s, \quad (3.4.23)$$

while the reduction of the divergence equation (3.4.20) yields

$$\partial_i \langle J_s^i \rangle_d = -\partial_i \left(\frac{\hat{u}_j}{\hat{T}} \right) \langle T_{diss}^{ij} \rangle_d - \left[\partial_i \left(\frac{\hat{\mu}}{\hat{T}} \right) - \frac{\hat{u}_k}{\hat{T}} F_{(0)i}^k \right] \langle J_{diss}^i \rangle_d, \quad (3.4.24)$$

which coincides with equation (2.19) of [63].

The reduced shear viscosity, $\hat{\eta}$, the heat conductivity, $\hat{\kappa}$ and the bulk viscosity $\hat{\zeta}_s$ can then be extracted from the formulæ:

$$\hat{P}_k^i \hat{P}_l^j \langle T_{ij}^{\text{diss}} \rangle_d - \frac{1}{d-1} \hat{P}_{kl} \hat{P}^{ij} \langle T_{ij}^{\text{diss}} \rangle_d = -2\hat{\eta} \hat{\sigma}_{kl}, \quad (3.4.25)$$

$$\hat{P}_i^j \left(\langle J_j^{\text{diss}} \rangle_d + \frac{\hat{q}}{\hat{\epsilon} + \hat{P}} \hat{u}^i \langle T_{ij}^{\text{diss}} \rangle_d \right) = -\hat{\kappa} \left(\hat{P}_{ij} \partial^j \frac{\hat{\mu}}{\hat{T}} + \frac{F_{(0)ij} \hat{u}^j}{\hat{T}} \right), \quad (3.4.26)$$

$$\frac{\hat{P}^{ij} \langle T_{ij}^{\text{diss}} \rangle_d}{d-1} - \frac{\partial \hat{P}}{\partial \hat{\epsilon}} \hat{u}^i \hat{u}^j \langle T_{ij}^{\text{diss}} \rangle_d + \frac{\partial \hat{P}}{\partial \hat{q}} \hat{u}^i \langle J_i^{\text{diss}} \rangle_d = -\hat{\zeta}_s \partial_i \hat{u}^i. \quad (3.4.27)$$

¹²Note that our conventions relate to those of [63] by changing $\hat{A}_{(0)i} \rightarrow -\hat{A}_{(0)i}$ and consequently $F_{(0)}^{ij} \rightarrow -F_{(0)}^{ij}$. This has no impact on (3.4.25) or (3.4.27), but changes the relative signs in (3.4.26) as well as in the conservation equation for the reduced boundary stress-energy tensor (3.2.17).

Using (3.4.18) the last two become

$$\hat{P}^{ij} \langle J_j^{\text{diss}} \rangle_d \left(1 - \frac{\hat{q}}{\hat{\epsilon} + \hat{P}} \tanh \omega \right) = -\hat{\kappa} \left(\hat{P}^{ij} \partial_j \frac{\hat{\mu}}{\hat{T}} + \frac{F_{(0)}^{ij} \hat{u}_j}{\hat{T}} \right) \quad (3.4.28)$$

$$\frac{\hat{P}^{ij} \langle T_{ij}^{\text{diss}} \rangle_d}{d-1} + \left(\frac{\partial \hat{P}}{\partial \hat{\epsilon}} \tanh^2 \omega + \frac{\partial \hat{P}}{\partial \hat{q}} \tanh \omega \right) \langle \mathcal{O}_1^{\text{diss}} \rangle_d = -\hat{\zeta}_s \partial_i \hat{u}^i. \quad (3.4.29)$$

Using the conservation equations for the fluid:

$$\partial_i \langle T^{ij} \rangle_d = F_{(0)}^{ij} \langle J_i \rangle_d, \quad (3.4.30)$$

$$\partial^i \langle J_i \rangle_d = 0, \quad (3.4.31)$$

yields:

$$\partial_j \log m = \frac{\cosh \omega}{\sinh \omega} \hat{u} \cdot \partial \omega \hat{u}_j - \cosh^2 \omega \hat{u} \cdot \partial \hat{u}_j + \sinh \omega \cosh \omega \hat{u}^i F_{(0)ij}, \quad (3.4.32)$$

$$\hat{u} \cdot \partial \omega = \frac{\sinh \omega \cosh \omega}{2(\sigma-1) \cosh^2 \omega + 1} \partial \cdot \hat{u}. \quad (3.4.33)$$

We also calculate

$$\begin{aligned} \langle T_{ij}^{\text{diss}} \rangle_d &= -2\eta_d \left[\cosh \omega \hat{\sigma}_{ij} + \sinh \omega \hat{u}_{(i} \left(\partial_{j)} \omega + \frac{1}{2} \sinh 2\omega \hat{u} \cdot \partial \hat{u}_{j)} - \cosh^2 \omega \hat{u}_k F_{(0)j)}^k \right) \right. \\ &\quad \left. + \cosh \omega \frac{\hat{P}_{ij}}{d-1} \left(1 - \frac{(d-1) \cosh^2 \omega}{2(\sigma-1) \cosh^2 \omega + 1} \right) \partial \cdot \hat{u} \right], \end{aligned} \quad (3.4.34)$$

$$\langle J_j^{\text{diss}} \rangle_d = \eta_d \cosh \omega \left[\hat{u}_j \hat{u} \cdot \partial \omega - \partial_j \omega - \sinh \omega \cosh \omega \hat{u} \cdot \partial \hat{u}_j - \cosh^2 \omega \hat{u}_i F_{(0)j}^i \right], \quad (3.4.35)$$

$$\langle \mathcal{O}_1^{\text{diss}} \rangle_d = \langle \mathcal{O}_2^{\text{diss}} \rangle_d = \langle \mathcal{O}_\psi^{\text{diss}} \rangle_d = 2\eta_d \frac{\cosh^2 \omega}{\sinh \omega} \hat{u} \cdot \partial \omega, \quad (3.4.36)$$

$$\langle \mathcal{O}_\zeta^{\text{diss}} \rangle_d = 0, \quad (3.4.37)$$

so that finally

$$\hat{\eta} = \eta_d \cosh \omega = L m^{2\sigma-1} \cosh \omega, \quad (3.4.38)$$

$$\hat{\kappa} = \frac{\eta_d \hat{T}}{\cosh \omega} = \frac{\sigma L m^{2\sigma}}{2\pi \cosh^2 \omega}, \quad (3.4.39)$$

$$\hat{\zeta}_s = \frac{2\eta_d \cosh \omega}{2\sigma-1} \left[\frac{2\sigma-d}{d-1} - \frac{2 \sinh^2 \omega ((\sigma-1) \cosh^2 \omega + \sigma)}{(2(\sigma-1) \cosh^2 \omega + 1)^2} \right]. \quad (3.4.40)$$

η_d is the shear viscosity of the (reduced) neutral case

$$\eta_d = \frac{L}{L_{\text{AdS}}} \eta_{2\sigma} = L m^{2\sigma-1}, \quad (3.4.41)$$

where the first equality comes from the reduction, while in the second equality we used the universal value of $\eta_{2\sigma}$ for conformal, AdS black branes, (3.4.4).

Note that the transport coefficients (3.4.38)-(3.4.39)-(3.4.40) are the universal coefficients valid for any solution with the same asymptotics as the black brane solution discussed in the previous section. Using (3.3.20) and (3.4.38), we find that $\hat{\eta}/\hat{s} = 1/4\pi$ and the KSS bound [64] is saturated for charged branes, as a consequence of the fact that it is saturated for the conformal branes, similar to the neutral case [25]. This is a dynamical statement: the value of η/s is fixed by the requirements of regularity in the interior (singular solutions can have η/s smaller or larger than $1/4\pi$).

We now turn to the ratio $\hat{\zeta}_s/\hat{\eta}$. As discussed in [25], in this ratio the factor $\eta_{2\sigma}$ drops out and the value of the ratio is fixed kinematically by the reduction: any solution with the given asymptotics, regular or singular, will have the same ratio. The same comment applies to the ratio $\hat{\kappa}/\hat{\eta}$. Our result for $\hat{\zeta}_s/\hat{\eta}$ can be compared with a recent formula in [69]:

$$\frac{\hat{\zeta}_s}{\hat{\eta}} = \sum_i \left(\hat{s} \frac{d\phi_i^h}{d\hat{s}} + \hat{q}_a \frac{d\phi_i^h}{d\hat{q}_a} \right)^2, \quad (3.4.42)$$

where \hat{q}_a are conserved charge densities and ϕ_i^h are a collection of scalar fields, evaluated at the event horizon, and the formula is valid in the Einstein frame where the entropy density s is given by the quarter of the horizon area. This formula reproduces all known results and we would like to check it against our result (3.4.40).

The entropy and charge density in the Einstein frame are still given by (3.3.20) and (3.3.18) from which it is straightforward to derive

$$\begin{aligned} d(\log \hat{s})|_{\hat{q}} &= -\frac{2(\sigma-1) \cosh^2 \omega + 1}{2\sigma \cosh \omega \sinh \omega} d\omega \\ d(\log \hat{q})|_{\hat{s}} &= \frac{2(\sigma-1) \cosh^2 \omega + 1}{(2\sigma-1) \cosh \omega \sinh \omega} d\omega \\ d(\psi_h)|_{\hat{s}} &= \sqrt{\frac{2(d-1)}{(2\sigma-1)(2\sigma-d)}} \tanh \omega d\omega \\ d(\psi_h)|_{\hat{q}} &= -\sqrt{\frac{2(2\sigma-1)}{(d-1)(2\sigma-d)}} \frac{2(\sigma-d) \cosh^2 \omega + d}{2\sigma \cosh \omega \sinh \omega} d\omega \\ d(\zeta_h)|_{\hat{s}} &= d(\zeta_h)|_{\hat{q}} = \sqrt{\frac{2(2\sigma-d-1)}{(2\sigma-d)}} \tanh \omega d\omega \end{aligned} \quad (3.4.43)$$

so that grouping everything together in (3.4.42), one does recover (3.4.40). This constitutes a very non-trivial check, since the two methods are completely different.

Moreover, direct computation yields

$$\frac{\hat{\zeta}_s}{\hat{\eta}} = 2 \left(\frac{1}{d-1} - \hat{c}_s^2 \right) - \frac{4 \sinh^2 \omega ((\sigma-1) \cosh^2 \omega + 1)}{(2(\sigma-1) \cosh^2 \omega + 1)^2} \quad (3.4.44)$$

so that the bound proposed in [65]

$$\frac{\hat{\zeta}_s}{\hat{\eta}} \geq 2 \left(\frac{1}{d-1} - \hat{c}_s^2 \right) \quad (3.4.45)$$

is always violated¹³, except if

$$\sigma < \hat{\mu}^2. \quad (3.4.46)$$

This is possible only if $\sigma < 1$ since $\hat{\mu}^2 = \tanh^2 \omega \leq 1$ but for all values in (3.1.26) $\sigma > 1$. The equality is achieved when either $\hat{\mu} = 0$ (neutral case) or else $\hat{\mu}^2 = \sigma$. Let us emphasize again that the ratio $\hat{\zeta}_s/\hat{\eta}$ is fixed kinematically, given asymptotics, so there is no reason to expect that a general system would satisfy such an inequality. In the charged case the asymptotic behavior is different from the neutral one since the presence of a chemical potential (and regularity at the horizon) implies that a non-normalizable mode for the gauge field is turned on, see (3.3.24).

We note, however, that there is a similar looking inequality to (3.4.45) that is saturated by the neutral branes and is satisfied by the charged ones, namely (3.4.45) but with the adiabatic speed of sound \hat{c}_s^2 replaced by \hat{c}_q^2

$$\hat{c}_q^2 \equiv \left. \frac{\partial \hat{P}}{\partial \hat{\epsilon}} \right|_{\hat{q}} = \frac{\cosh 2\omega}{(2\sigma - 2) \cosh^2 \omega + 1}. \quad (3.4.47)$$

\hat{c}_q^2 reduces to the speed of sound of the conformal branes when $\omega = 0$ and furthermore

$$\frac{\hat{\zeta}_s}{\hat{\eta}} - 2 \left(\frac{1}{d-1} - \hat{c}_q^2 \right) = \frac{(\sigma - 1) \sinh^2(2\omega)}{(2(\sigma - 1) \cosh^2 \omega + 1)^2}. \quad (3.4.48)$$

The right hand side is manifestly positive when $\sigma > 1$, which is true for all values in (3.1.26). It would be interesting to check whether there are any counterexamples to this inequality.

The DC conductivity can be deduced using the Franz-Wiedemann law:

$$\hat{\sigma}_{DC} = \frac{\hat{\kappa}}{\hat{T}} = \frac{\eta_d}{\cosh \omega} = \frac{L m^{2\sigma-1}}{\cosh \omega}. \quad (3.4.49)$$

In order to make comparisons with other results easier, one can reexpress all the transport coefficients for the reduced AdS black brane in terms of the temperature and chemical potential:

$$\hat{\eta} = L \left(\frac{2\pi \hat{T}}{\sigma} \right)^{2\sigma-1} (1 - \hat{\mu}^2)^{-\sigma}, \quad (3.4.50)$$

$$\hat{\kappa} = \frac{\sigma L}{2\pi} \left(\frac{2\pi \hat{T}}{\sigma} \right)^{2\sigma} (1 - \hat{\mu}^2)^{1-\sigma}, \quad (3.4.51)$$

$$\hat{\zeta}_s = \frac{2(2\sigma - d)\hat{\eta}}{(d-1)(2\sigma-1)} \left[1 - \frac{2(d-1)\hat{\mu}^2(2\sigma-1-\sigma\hat{\mu}^2)}{(2\sigma-d)(2\sigma-1-\hat{\mu}^2)^2} \right], \quad (3.4.52)$$

$$\hat{\sigma}_{DC} = L \left(\frac{2\pi \hat{T}}{\sigma} \right)^{2\sigma-1} (1 - \hat{\mu}^2)^{1-\sigma}. \quad (3.4.53)$$

¹³See [66] for recent work containing other such examples.

Note that taking the neutral limit $\hat{\mu} \rightarrow 0$ in the DC conductivity gives a finite contribution: indeed our computation represents the microscopic fluctuations around the background, whether the latter is neutral or charged. The result for the conductivity can be compared with results from the flavour branes approach, [67, 68, 6, 14], but only in the zero density limit. Indeed, (3.4.53) is obtained by working out the fluctuations of the metric and gauge field around a charged black hole, while in the case of flavour branes, the gauge field lives on the brane in a neutral background and does not backreact¹⁴.

4. Discussion and conclusions

In this paper we set up holography for non-asymptotically AdS solutions of a class of Einstein-Maxwell-Dilaton theories. This was achieved by showing that these theories are related to AdS-Maxwell theory in higher dimensions by means of a generalized dimensional reduction over compact Einstein manifolds. ‘Generalized’ here refers to the continuation of the dimension of the compact space to non-integral values. Such a generalized dimensional reduction was introduced in [25] and here we include gauge fields and additional scalar fields in the analysis.

The relation to higher dimensional AdS gravity controls both the UV and the IR behavior of the strongly coupled dual QFT. The UV behavior is dictated by a fixed point at $d + \epsilon$ dimensions, where ϵ is the dimension of the compact space, whose existence follows from the fact that the solution uplifts to an asymptotically AdS solution. From the d -dimensional perspective this translates into specific running of coupling constants. The IR behavior near thermal equilibrium, the hydrodynamic regime, is also controlled by the higher-dimensional theory. The universal hydrodynamic behavior of AdS gravity implies via dimensional reduction a specific hydrodynamic behavior of EMD theories. In particular, an entropy current with non-negative divergence in AdS reduces to a entropy current with the same property in the reduced theory and the transport coefficients are directly related to those of AdS gravity [25].

This leads to certain kinematical relations among the transport coefficients. For example, the ratios of the bulk to shear viscosity and conductivity to shear viscosity are fixed to specific values irrespectively of whether the bulk solution is regular or singular in the interior. Furthermore, when there is a chemical potential the putative bound on the bulk to shear viscosity proposed in [65] is generically violated.

The duality described here is not in general valid at all energy scales. A prototype example for the dualities we discuss is the holographic duality for non-conformal branes. In that case, as discussed in detail in section 2 of [22], one assumes that the effective ‘t Hooft coupling $g_{eff}^2 N$ is fixed while N^2 is taken to infinity. However, in these theories the effective coupling constant depends on the energy scale so there is always a regime where $g_{eff}^2 N$ grows faster than N^2 implying that the dilaton blows up and a new description is needed (which for the case of Dp branes is typically that of an M-brane). Our current discussion is not tied to any specific dual theory but we expect the same to be true here: the holographic description would be valid below a certain energy scale.

¹⁴We wish to thank E. Kiritsis and F. Nitti for discussions on this point.

The recent interest in these theories originates from the desire to model holographically interesting IR fixed points, mostly having in mind applications to condensed matter systems. Models that interpolate between the IR behavior described here and an AdS region in the UV have been considered, for example, in [3, 6, 9, 11, 70, 71, 14]. One would expect on general grounds to be able to model the IR region without a reference to such UV completion and indeed our discussion provides precisely such a description.

There are many possible extensions and generalizations of this work. In section 2 we described an array of theories which are linked with AdS-Maxwell gravity in higher dimensions but we only worked out the holographic dictionary and the hydrodynamic regime for one of them. It would be interesting to work out the holographic data for the entire class. For example, the case of two gauge fields is interesting since such systems could be used to describe holographically imbalanced superconductors, see [72] for recent work in this direction. The case where the higher-dimensional theory is AdS-Gauss-Bonnet is under investigation, [73]. More generally, it would be interesting to map out all possibilities where such a generalized dimensional reduction could be used in order to set up holography.

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A. Appendix

A.1 Diagonal reduction of $(2\sigma + 1)$ -AdS-Maxwell theories

Our starting point is the Einstein-AdS action with a cosmological constant in $2\sigma + 1$ dimensions (2.1.1) and a Maxwell field strength:

$$S_{(2\sigma+1)} = \frac{1}{16\pi G_N^{(2\sigma+1)}} \int d^{2\sigma+1} \sqrt{-g_{(2\sigma+1)}} \left[R_{(2\sigma+1)} - \frac{1}{4} F^2 - 2\Lambda \right]. \quad (\text{A.1.1})$$

It has equations of motion

$$G_{AB} + \Lambda g_{AB} = \frac{1}{2} F_{AC} F_B{}^C - \frac{1}{8} F^2 g_{AB}, \quad (\text{A.1.2})$$

$$\nabla_A F^{AB} = 0. \quad (\text{A.1.3})$$

We wish to perform a reduction to an Einstein-Maxwell-Dilaton theory with a static Ansatz:

$$ds_{(2\sigma+1)}^2 = e^{2\alpha\phi} ds_{(d+1)}^2 + e^{2\beta\phi} dX_{(2\sigma-d)}^2, \quad A_A = (A_M(x^N), A_a = 0) \quad (\text{A.1.4})$$

where $dX_{(2\sigma-d)}^2$ is the metric of a $(2\sigma - d)$ -dimensional Einstein space, (2.0.3), with normalised curvature $\lambda_{(2\sigma-d)}$.

For a diagonal Ansatz, it is consistent to take all scalar fields along each reduced direction equal, see [53] for the generic (toroidal) case. Nonetheless, let us check that such an Ansatz is consistent by reducing Einstein's equations directly and writing out the action from which they derive.

Using the tetrad formalism, the higher-dimensional Einstein tensor $G_{AB}^{(2\sigma+1)}$ can be projected on the external and internal coordinates:

$$\begin{aligned} G_{MN}^{(2\sigma+1)} &= G_{MN}^{(d+1)} + [(d-1)\alpha^2 + (2\sigma-d)2\alpha\beta - (2\sigma-d)\beta^2] \partial_M \phi \partial_N \phi - \\ &\quad - [(d-1)\alpha + (2\sigma-d)\beta] \nabla_M \nabla_N \phi - \\ &\quad - \frac{1}{2} g_{MN}^{(d+1)} \left\{ R_{(2\sigma-d)} e^{2(\alpha-\beta)\phi} - 2[(d-1)\alpha + (2\sigma-d)\beta] \square \phi - \right. \\ &\quad \left. - [(d-1)(d-2)\alpha^2 + 2(2\sigma-d)(d-2)\alpha\beta + (2\sigma-d)(2\sigma-d+1)\beta^2] \partial\phi^2 \right\} \end{aligned} \quad (\text{A.1.5})$$

$$\begin{aligned} G_{ab}^{(2\sigma+1)} &= G_{ab}^{(2\sigma-d)} - \frac{1}{2} g_{ab}^{(2\sigma-d)} e^{2(\beta-\alpha)\phi} \left\{ R_{(d+1)} - 2[d\alpha + (2\sigma-d-1)] \square \phi - \right. \\ &\quad \left. - [d(d-1)\alpha^2 + 2(d-1)(2\sigma-d-1)\alpha\beta + (2\sigma-d)(2\sigma-d-1)\beta^2] \partial\phi^2 \right\} \end{aligned} \quad (\text{A.1.6})$$

where $G_{MN}^{(d+1)}$ and $G_{ab}^{(2\sigma-d)}$ are respectively the Einstein tensor associated to the $(d+1)$ -dimensional metric and to the $(2\sigma-d)$ -dimensional compact space $X_{(2\sigma-d)}$. Then, taking the trace of $G_{AB}^{(2\sigma+1)}$, one finds the Ricci scalar

$$\begin{aligned} R_{(2\sigma+1)} e^{2\alpha\phi} &= R_{(d+1)} + e^{2(\alpha-\beta)\phi} X_{(2\sigma-d)} - 2(d\alpha + (2\sigma-d)\beta) \square \phi - \\ &\quad - [d(d-1)\alpha^2 + (2\sigma-d)(2\sigma-d+1)\beta^2 + 2(2\sigma-d)(d-1)\alpha\beta] \partial\phi^2, \end{aligned} \quad (\text{A.1.7})$$

while

$$\det g_{(2\sigma+1)} = e^{[2(d+1)a + (2\sigma-d)b]\phi} \det g_{(d+1)}. \quad (\text{A.1.8})$$

Setting the overall conformal factor in ϕ in the action to unity¹⁵ requires

$$(2\sigma-d)\beta = (1-d)\alpha \quad (\text{A.1.9})$$

upon which

$$R_{(2\sigma+1)} e^{2\alpha\phi} = R_{(d+1)} - 2\alpha \square \phi - (d-1) \frac{2\sigma-1}{2\sigma-d} \alpha^2 \partial\phi^2 + e^{2\frac{2\sigma-1}{2\sigma-d}\alpha\phi} R_{(2\sigma-d)}. \quad (\text{A.1.10})$$

In order to have a canonically normalised kinetic term for the scalar, we then set

$$\alpha = -\sqrt{\frac{(2\sigma-d)}{2(d-1)(2\sigma-1)}} = -\frac{\delta}{2} \quad \Leftrightarrow \quad \delta = \sqrt{\frac{2(2\sigma-d)}{(d-1)(2\sigma-1)}} \quad (\text{A.1.11})$$

so that the bulk action naively becomes

$$\begin{aligned} S_{(d+1)} &= \frac{1}{16\pi G_N^{(d+1)}} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g_{(d+1)}} \left[R_{(d+1)} - \frac{1}{2} \partial\phi^2 - \frac{1}{4} e^{\delta\phi} F^2 - 2\Lambda e^{-\delta\phi} + \right. \\ &\quad \left. + R_{(2\sigma-d)} e^{-\frac{2\phi}{(d-1)\delta}} \right] - \frac{1}{16\pi G_N^{(d+1)}} \int_{\partial\mathcal{M}} d^d x \sqrt{-h_{(d)}} \delta n^M \partial_M \phi. \end{aligned} \quad (\text{A.1.12})$$

¹⁵ *e.g.*, going to the Einstein frame.

To check that this is correct, we can also replace in (A.1.6) and (A.1.7)

$$G_{MN}^{(2\sigma+1)} = G_{MN}^{(d+1)} - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{1}{2}g_{MN}^{(d+1)}\left[R_{(2\sigma-d)}e^{-\frac{2\phi}{(d-1)\delta}} - \frac{1}{2}\partial\phi^2\right] \quad (\text{A.1.13})$$

$$G_{ab}^{(2\sigma+1)} = G_{ab}^{(2\sigma-d)} - \frac{1}{2}g_{ab}^{(2\sigma-d)}e^{\frac{2\phi}{(d-1)\delta}}\left[R_{(d+1)} + \frac{2}{(d-1)\delta}\square\phi - \frac{1}{2}\partial\phi^2\right]. \quad (\text{A.1.14})$$

and reexpress Einstein's equations (A.1.2):

$$\begin{aligned} G_{MN}^{(d+1)} &= \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{e^{\delta\phi}}{2}F_{MP}F_M^P + \\ &+ \frac{g_{MN}^{(d+1)}}{2}\left[R_{(2\sigma-d)}e^{-\frac{2\phi}{(d-1)\delta}} - \frac{1}{2}\partial\phi^2 - \frac{e^{\delta\phi}}{4}F^2 - 2\Lambda e^{-\delta\phi}\right] \end{aligned} \quad (\text{A.1.15})$$

$$G_{ab}^{(2\sigma-d)} = \frac{g_{ab}^{(2\sigma-d)}}{2}e^{\frac{2\phi}{(d-1)\delta}}\left[R_{(d+1)} + \frac{2\square\phi}{(d-1)\delta} - \frac{1}{2}\partial\phi^2 - \frac{e^{\delta\phi}}{4}F^2 - 2\Lambda e^{-\delta\phi}\right] \quad (\text{A.1.16})$$

In (A.1.15), we recognise the lower-dimensional equation of motion for the metric, as derived from (A.1.12). Taking the trace of (A.1.16) and replacing again in (A.1.16), one finds that $\mathbf{X}_{(2\sigma-d)}$ must be an Einstein space, that is

$$R_{ab}^{(2\sigma-d)} = \frac{R^{(2\sigma-d)}}{2\sigma-d}g_{ab}^{(2\sigma-d)}. \quad (\text{A.1.17})$$

The lower-dimensional Ricci scalar can be derived from (A.1.15) or (A.1.16):

$$R_{(d+1)} = \frac{1}{2}\partial\phi^2 + \frac{(d-3)e^{\delta\phi}}{4(d-1)}F^2 + \frac{d+1}{d-1}2\Lambda e^{-\delta\phi} - \frac{d+1}{d-1}R_{(2\sigma-d)}e^{-\frac{2\phi}{(d-1)\delta}} \quad (\text{A.1.18})$$

$$R_{(d+1)} = \frac{1}{2}\partial\phi^2 + \frac{e^{\delta\phi}}{4}F^2 + 2\Lambda e^{-\delta\phi} - \frac{2\sigma-d-2}{2\sigma-d}R_{(2\sigma-d)}e^{-\frac{2\phi}{(d-1)\delta}} - \frac{2\square\phi}{(d-1)\delta} \quad (\text{A.1.19})$$

Subtracting the two previous equations yields the dilaton equation of motion:

$$\square\phi = \frac{\delta}{4}e^{\delta\phi}F^2 - 2\delta\Lambda e^{-\delta\phi} + \frac{2}{(d-1)\delta}R_{(2\sigma-d)}e^{-\frac{2\phi}{(d-1)\delta}}, \quad (\text{A.1.20})$$

identical to that derived from (A.1.12), while the other combination gives back the trace of Einstein's equations. Finally, the lower-dimensional Maxwell equation follows straightforwardly from the higher-dimensional one (A.1.3).

The metric Ansatz becomes

$$ds_{(2\sigma+1)}^2 = e^{-\delta\phi}ds_{(d+1)}^2 + e^{\frac{\phi}{\delta}(\frac{2}{d-1}-\delta^2)}dX_{(2\sigma-d)}^2. \quad (\text{A.1.21})$$

We have also defined the lower-dimensional Newton's constant $G_N^{(d+1)} = G_N^{(2\sigma+1)}/V_{(2\sigma-d)}$, where $V_{(2\sigma-d)}$ is the volume of $\mathbf{X}^{(2\sigma-d)}$. The term in $\square\phi$ does not impact the lower-dimensional equations generates a boundary term on $\partial\mathcal{M}$, the boundary of the bulk space-time \mathcal{M} defined by its unit normal vector n^μ and boundary metric $h_{(d)}$. It has no impact on the equations of motion, but would be important for the computation of the Euclidean action on-shell.

Let us also consider the reduction of the Gibbons-Hawking-York boundary term, which involves the trace $K_{(2\sigma)}$ of the extrinsic curvature of spacetime. Using the Ansatz (A.1.21), it is a matter of calculation to show that

$$\sqrt{-h_{(2\sigma)}} = e^{-\frac{\delta}{2}\phi} \sqrt{-h_{(d)}}, \quad K_{(2\sigma)} = e^{\frac{\delta}{2}\phi} \left[K_{(d)} - \frac{\delta}{2} n^M \partial_M \phi \right]. \quad (\text{A.1.22})$$

We can move on and deal with the Gibbons-Hawking-York boundary term:

$$\begin{aligned} S_{(2\sigma)}^{GBH} &= -\frac{1}{8\pi G_N^{(2\sigma+1)}} \int_{\partial\mathcal{M}} \sqrt{-h_{(2\sigma)}} d^{2\sigma}x K_{(2\sigma)} \\ &= -\frac{1}{8\pi G_N^{(d+1)}} \int_{\partial\mathcal{M}} \sqrt{-h_{(d)}} d^{2\sigma}x \left[K_{(d)} - \frac{\delta}{2} n^M \partial_M \phi \right], \end{aligned} \quad (\text{A.1.23})$$

where we have used (A.1.22). The first term is the lower-dimensional Gibbons-Hawking-York boundary term, while the second term is exactly the one needed so that the boundary term coming from the reduction of the higher-dimensional Ricci scalar is cancelled, see (A.1.12). In the end, only the $(d+1)$ -dimensional GHY term is left.

Let us now make contact with the generic Einstein-Dilaton action (2.1.4)

$$\begin{aligned} S_{(d+1)} &= \frac{1}{16\pi G_N^{(d+1)}} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\gamma\phi}F^2 - 2\Lambda_1 e^{-\delta_1\phi} - 2\Lambda_2 e^{-\delta_2\phi} \right] - \\ &\quad - \frac{1}{8\pi G_N^{(d+1)}} \int_{\partial\mathcal{M}} \sqrt{-h_{(d)}} d^{2\sigma}x K_{(d)}, \end{aligned} \quad (\text{A.1.24})$$

As shown above, using the metric Ansatz:

$$ds_{(2\sigma+1)}^2 = e^{-\delta_1\phi} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta_1}(\frac{2}{d-1}-\delta_1^2)} dX_{(2\sigma-d)}^2, \quad (\text{A.1.25})$$

and setting

$$\Lambda_1 = \Lambda, \quad R_{(2\sigma-d)} = -2\Lambda_2, \quad \delta_2 = \frac{2}{(d-1)\delta_1} \quad (\text{A.1.26})$$

so that the first Liouville potential in (2.1.4) originates from the higher-dimensional cosmological constant Λ and the second one from the curvature of the internal space, this action is a consistent reduction of the Einstein-AdS-Maxwell action. The exponent δ_1 is related to the number of reduced dimensions as:

$$\delta_1 = \sqrt{\frac{2(2\sigma-d)}{(d-1)(2\sigma-1)}} \Leftrightarrow 2\sigma = \frac{d-(d-1)\delta_1^2}{2-(d-1)\delta_1^2} \quad (\text{A.1.27})$$

from which 2σ vary with δ_1 in the following way:

A consistent range of dimension values for the higher-dimensional theory is $0 \leq \delta_1^2 \leq 2/(d-1)$.

To extend δ_1 to the complementary range, let us reverse the origins of the Liouville potentials in (A.1.24), whereupon δ_1 has to be set to

$$\delta_1^2 = \frac{2(2\sigma-1)}{(d-1)(2\sigma-d)} \Leftrightarrow 2\sigma = \frac{d-(d-1)\delta_1^2-2}{(d-1)\delta_1^2-2}, \quad (\text{A.1.28})$$

δ_1^2	0^+	$\left(\frac{2}{d-1}\right)^-$	$\left(\frac{2}{d-1}\right)^+$	$\left(\frac{2d}{d-1}\right)^-$	$\left(\frac{2d}{d-1}\right)^+$	$+\infty$
$2\sigma + 1$	$(d+1)^+$	\nearrow	$+\infty$	\nearrow	0^-	$(d+1)^-$

δ_1^2	0^+	$\left(\frac{2}{d-1}\right)^-$	$\left(\frac{2}{d-1}\right)^+$	$\left(\frac{2d}{d-1}\right)^-$	$\left(\frac{2d}{d-1}\right)^+$	$+\infty$
$2\sigma + 1$	2^-	\searrow	$+\infty$	\searrow	$(d+2)^-$	$(d+1)^+$

with the Λ_1 Liouville now descending from the curvature of the internal space $R_{(2\sigma-d)}$, the Λ_2 one from the higher-dimensional constant Λ . The range of dimension values spanned by δ is now:

Its consistent restriction $\delta_1^2 > 2/(d-1)$ is indeed the complementary of the previous one. Note that there is an upper bound on δ_1 ,

$$\delta_1^2 < \delta_{max}^2 = \frac{2d}{d-1} \quad (\text{A.1.29})$$

which reflects the fact that the space $\mathbf{X}_{(2\sigma-d)}$ only has non-zero curvature if $2\sigma > d+2$.

The metric Ansatz is:

$$ds_{(2\sigma+1)}^2 = e^{-\frac{2\phi}{(d-1)\delta_1}} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta_1}(\delta_1^2 - \frac{2}{d-1})} dX_{(2\sigma-d)}^2. \quad (\text{A.1.30})$$

The higher-dimensional theory is still Einstein-AdS but the inclusion of a Maxwell field in the higher-dimensional action requires $\gamma = \frac{2}{(d-1)\delta_1}$ in the lower-dimensional EMD action.

A.2 Non-diagonal reduction of AdS theories along an \mathbf{S}^1

Let us start again from the Einstein-AdS theory (2.1.1). We then reduce along a circle \mathbf{S}^1 , this time by means of a non-diagonal Ansatz

$$ds_{(d+2)}^2 = e^{2\alpha\phi} ds_{(d+1)}^2 + e^{-2(d-1)\alpha\phi} (dy + \mathcal{A})^2, \quad 2\sigma = d+1. \quad (\text{A.2.1})$$

with

$$\mathcal{A} = \mathcal{A}_M dx^M. \quad (\text{A.2.2})$$

We use calligraphic notation to distinguish gauge fields arising in the $(d+1)$ -dimensional theory through the compactification from those descending from higher-dimensional ones. Then,

$$e^{2\alpha\phi} R_{(d+2)} = R_{(d+1)} - d(d-1)\alpha^2 \partial\phi^2 - \frac{1}{4} e^{-2d\alpha\phi} \mathcal{F}_{MN} \mathcal{F}^{MN} \quad (\text{A.2.3})$$

discarding the $O(\square\phi)$ term at this point. Normalising the kinetic term for the scalar field automatically gives

$$\alpha = -\frac{1}{\sqrt{2d(d-1)}} \quad (\text{A.2.4})$$

and setting

$$\delta = \sqrt{\frac{2}{d(d-1)}}, \quad \gamma = \sqrt{\frac{2d}{d-1}}, \quad \gamma\delta = \frac{2}{d-1} \quad (\text{A.2.5})$$

we recover the EMD action (2.2.2)

$$S_{(d+1)} = \frac{1}{16\pi G_N^{(d+1)}} \int d^{d+1}x \sqrt{-g_{(d+1)}} \left[R_{(d+1)} - \frac{1}{2} \partial\phi^2 - \frac{1}{4} e^{\frac{2\phi}{(d-1)\delta}} \mathcal{F}_{\mu\nu} \mathcal{F}^{MN} - 2\Lambda e^{-\delta\phi} \right], \quad (\text{A.2.6})$$

albeit with a single Liouville potential: reducing over an \mathbf{S}^1 does not generate a potential due to its zero curvature. The metric Ansatz becomes:

$$ds_{(d+2)}^2 = e^{-\delta\phi} ds_{(d+1)}^2 + e^{\frac{\phi}{\delta}(\frac{2}{d-1}-\delta^2)} (dy + \mathcal{A})^2. \quad (\text{A.2.7})$$

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